

The Retroazimuthal Problem

Fixed-Direction Map Projections and the Historical Case of the Qibla

A self-contained, stand-alone volume in the tradition of the cartographic distortion series

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Preface and Status of This Volume

The earlier volumes of the cartographic distortion series treat projections chosen for their general historical or computational importance (azimuthal projections, cylindrical projections, discrete global grid systems). This volume instead treats a single family of projections defined by one precise geometric property — retroazimuthality — and follows it from the ritual motivation that generated it (the determination of the qibla direction in medieval Islamic astronomy) to the three historical cartographic constructions that solve it, each at a different price: Littrow (1833) achieves conformality by giving up freedom in the choice of centre; Craig (1909) and Hammer (1910) achieve exactness for an arbitrary centre by giving up conformality and, in Hammer’s case, global injectivity.

Unlike the fifth volume of the series (on discrete global grid systems), still an open, continuously expanding construction site, this volume was conceived from the outset as self-contained and complete in itself, originally at eleven chapters plus one appendix. It has since grown by three further chapters. The first ([Chapter 14](#)) was added not to reopen the volume’s scope but to *test* it: the closed-form construction and the exact bearing formula derived and used throughout are generic in their centre parameter φ_C, λ_C , and [Chapter 14](#) checks this genericity against real documented centres other than Mecca, rather than leaving it an unexercised claim — now extended a third time, to a fourth centre (the Samaritan Mount Gerizim orientation), alongside Jerusalem and Bahj . The second ([Chapter 7](#)) closes, as far as this volume’s resources allow, one of the items [Chapter 15](#) had flagged as open: it derives a rigorous local tangent-plane approximation of the exact bearing formula, uses it to resolve a genuine subtraction-order ambiguity in the secondary report of al-Battani’s own approximate method, and quantifies, for the first time in this volume, how that method’s accuracy actually degrades with distance from Mecca. The third ([Chapter 5](#)) takes up another item [Chapter 4](#) had left open: consulting King (1997) and Dekker (2000) directly, rather than through Tobler’s (2002) paraphrase or this volume’s own earlier secondary account of the two scholars’ dispute over a possible Islamic precursor of the retroazimuthal idea. As far as it can be done without the full text of either paper, it finds King’s own antiquity claim more hedged, and Dekker’s own objection narrower and more agnostic, than either had appeared secondhand — and surfaces one previously unrecorded detail, a third construction (Schoy’s variant) alongside Craig’s and Hammer’s. A second, practical appendix ([Appendix B](#)) was also added, extending the original appendix’s forty-four-city qibla reference table to the three further centres of [Chapter 14](#), with the same coordinates and the same formulas, so that the great-circle/rhumb-line divergence documented there on a seven-city test set can be inspected at full geographic scale. The volume, now fourteen chapters plus two appendices, is not expected to grow further. As stated explicitly in [Chapter 15](#), a number of questions remain flagged as open (*todoboxes* scattered through the text) — not because the volume is incomplete in its ambition, but because the discipline of not inventing data or unverified proofs requires flagging them as such rather than papering over them artificially.

Methodological Note

Every quantitative claim in this volume traces back either to a source cited in [Appendix B.3](#) or to a computation executed and verifiable within the volume itself (scripts in `code/`, results in JSON format alongside the scripts, referenced throughout the text and gathered

together in the practical reference table of [Appendix A](#)). Sections not yet supported by a primary source or an executed computation are flagged with a «To Complete / Verify» box and should not be read as settled results. At several points (notably [Chapters 4](#) and [13](#)) the volume explicitly flags the limits of the secondary sources available for its writing, carefully distinguishing claims that are proved from claims that are attributed and not independently re-verified.

Part I

Foundations

Chapter 1

Introduction: The Retroazimuthal Problem

1.1 Two problems, not one

Classical mathematical cartography — the subject of the four volumes of *No Map Is Best* and of the fifth volume on discrete global grid systems — almost always deals with a single geometric problem: given a projection $f : \mathbb{S}^2 \rightarrow \mathbb{R}^2$, how do angle, area, or distance distort in a neighbourhood of a point? The «azimuthality» of azimuthal projections, in this framework, is almost always understood in one direction only: the azimuthal equidistant projection centred at C preserves the bearing *from* C to every other point P . Whoever stands at C can read on the map, for every P , the exact bearing to follow in order to reach it.

But there is a distinct, less well known, and structurally harder geometric problem: preserving the bearing *toward* a fixed point C from *every* other point P on the map. Whoever stands at P , whatever P is, must be able to read on the map the direction in which to look in order to face C — not the other way round. This is the **retroazimuthal problem**, and it is far from equivalent to the first.

Definition 1.1 (Azimuthal vs. retroazimuthal projection). Let $f : \mathbb{S}^2 \setminus \{C\} \rightarrow \mathbb{R}^2$ be a projection with fixed reference point C . We say f is:

- **azimuthal at C** if, for every P , the angle that the segment $f(C)f(P)$ makes with cartographic north at $f(C)$ equals the great-circle bearing from C to P ;
- **retroazimuthal with respect to C** if, for every P , the angle that the segment $f(P)f(C)$ makes with cartographic north at $f(P)$ equals the great-circle bearing from P to C .

The difference is not cosmetic. On the sphere, because of the non-symmetry of bearing under exchange of the two points (the bearing from P to C is not, in general, the supplement or the opposite of the bearing from C to P , except in the degenerate Euclidean-plane case), a single family of projections cannot in general solve both problems at once with the same elementary construction. Azimuthal equidistant, gnomonic, stereographic, and similar projections all solve the first problem quite naturally: the centre of projection and the point of observation coincide by construction. The second problem requires a distinct geometric construction, with equations not reducible in any elementary way to those of standard azimuthal projections.

1.2 Why this problem was posed: the case of the qibla

The retroazimuthal problem does not arise as an abstract mathematical curiosity. It arises from a precise and extremely widespread practical need: determining, from any locality in the Islamic world, the direction of the Kaaba in Mecca, toward which ritual prayer (*qibla*) is

directed. From the eighth century onward Muslim astronomers tackled this problem with increasingly sophisticated spherical-geometry methods ([Chapter 4](#)); in the twentieth century the problem was reformulated in explicitly cartographic terms: construct a map from which anyone, anywhere, can read directly — measuring an angle from north with a ruler — the direction of Mecca.

The two classical solutions to this cartographic problem, both published in 1909–1910, are due to James Ireland Craig, a Scottish surveyor active in Egypt ([Chapter 9](#)), and to Ernst Hammer ([Chapter 10](#)). A third projection, seventy years earlier and due to Joseph Johann von Littrow (1833), solves a special case of the problem conformally — the only known conformal solution ([Chapter 8](#)) — but, as will be shown by an explicit numerical test, only for a fixed point tied to the equator of the chosen coordinate system, not for an arbitrary point such as Mecca.

Methodological Note

This volume treats the retroazimuthal problem as an autonomous mathematical object, of which the Islamic qibla is the historical case that motivated its study and provides its most natural and best-documented example. The volume is not, properly speaking, a treatise on the history of Islamic science: for that, the essential reference remains the work of David A. King, used here with due philological caution and without any claim to completeness ([Chapter 4](#)). Every projection formula reported has been checked against an explicitly cited reference implementation and against a computation executed for this volume (folder `code/`); no numerical value has been invented.

1.3 Structure of the volume

The volume is organised in four parts:

- **Part I — Foundations** ([Chapters 1 to 3](#)): the spherical-trigonometry background needed, a formal definition of the retroazimuthal problem, and a first non-coincidence result.
- **Part II — The historical case: the qibla** ([Chapters 4 and 6](#)): the science of the qibla in medieval Islamic astronomy, and a quantitative sensitivity analysis of the exact formula to geographic errors historically documented in the Ptolemaic-Islamic tradition.
- **Part III — The projections** ([Chapters 8 to 11](#)): the three historical constructions — Littrow, Craig, Hammer — with an explicit numerical check of the retroazimuthal property (and of its limits) for each, followed by a systematic comparison of residual distortion.
- **Part IV — Synthesis** ([Chapters 13 and 15](#)): a more formal discussion of why no global retroazimuthal projection can be conformal, a four-axis synthesis in the style of the series, and open questions.

Key Result

The thread running through this volume is that the retroazimuthal problem is not a minor variant of the azimuthal problem, but a geometrically distinct problem whose exact, global solution is incompatible with conformality: the historical constructions (Craig, Hammer) achieve global retroazimuthal exactness by giving up conformality and, in Hammer's case, injectivity; the only known conformal construction (Littrow) achieves conformality by giving up freedom of choice of centre, which remains tied to the equator of the coordinate system employed.

Chapter 2

Spherical Trigonometry Preliminaries

2.1 Conventions

We write $\varphi \in [-\pi/2, \pi/2]$ for latitude and $\lambda \in (-\pi, \pi]$ for longitude on the unit sphere $\mathbb{S}^2$ (the Earth's radius R is a global scale factor that does not alter any of the properties discussed here, and is omitted or set to $R = 1$ unless stated otherwise). A point $P = (\varphi, \lambda)$ determines, on the sphere, the vector

$$\mathbf{p} = (\cos \varphi \cos \lambda, \cos \varphi \sin \lambda, \sin \varphi).$$

Definition 2.1 (Great-circle distance and bearing). Given two points $P = (\varphi_1, \lambda_1)$ and $Q = (\varphi_2, \lambda_2)$, let $\Delta\lambda = \lambda_2 - \lambda_1$. The **great-circle angular distance** c between P and Q satisfies

$$\cos c = \sin \varphi_1 \sin \varphi_2 + \cos \varphi_1 \cos \varphi_2 \cos \Delta\lambda, \quad (2.1)$$

and the **initial bearing** (azimuth, measured from north, clockwise) from P to Q is given by

$$\theta_{P \rightarrow Q} = \text{atan2}(\sin \Delta\lambda \cos \varphi_2, \cos \varphi_1 \sin \varphi_2 - \sin \varphi_1 \cos \varphi_2 \cos \Delta\lambda). \quad (2.2)$$

This is the standard formula for the inverse geodetic problem on the sphere (see e.g. Snyder, *Map Projections: A Working Manual*, USGS PP 1395, ch. 4; the same form, specialised to $Q = \text{Mecca}$, is the one used in the literature on the science of the qibla, cf. [Chapter 4](#)).

Remark 2.2 (Non-symmetry of bearing). In general $\theta_{P \rightarrow Q} \neq \theta_{Q \rightarrow P} \pm \pi$. The equality $\theta_{Q \rightarrow P} = \theta_{P \rightarrow Q} + \pi \pmod{2\pi}$ always holds in the Euclidean plane (where bearings are defined relative to a single global «north» direction, i.e. a parallel vector field), but on the sphere parallel transport of the vector «north» along a geodesic does not return it to the same relative direction: it is precisely the positive Gaussian curvature of the sphere — already the protagonist of the Theorema Egregium in the introductory volume of the series — that breaks this symmetry. It is this symmetry breaking, quantifiable in terms of the area of the spherical triangle bounded by P , Q and the North Pole (Girard's formula), that makes the retroazimuthal problem structurally different from the azimuthal one.

2.2 Quantifying the asymmetry: the signed spherical excess

Remark 2.2 states qualitatively that the failure of $\theta_{Q \rightarrow P} = \theta_{P \rightarrow Q} + \pi$ is governed by the curvature captured in the polar triangle P – Q – N (with N the North Pole). We now make this quantitative.

Definition 2.3 (Signed spherical excess of the polar triangle). Let $E \geq 0$ be the (unsigned) spherical excess of the spherical triangle with vertices P , Q , N , i.e. $E = A + B + C - \pi$ where A, B, C are its three interior angles (all in $[0, \pi]$); by Girard's theorem E equals the area of the triangle on the unit sphere. Let $\Delta\lambda = \text{wrap}_{(-\pi, \pi]}(\lambda_Q - \lambda_P)$. Define the **signed spherical excess**

$$\tilde{E} = \text{sgn}(\Delta\lambda) \cdot E, \quad (2.3)$$

positive when Q lies east of P (equivalently, when N sits to the left of the directed geodesic $P \rightarrow Q$) and negative when Q lies west of P ; $\tilde{E} = 0$ when P and Q share a meridian (the triangle degenerates).

Proposition 2.4 (Bearing-reversal discrepancy). *For any two non-antipodal points P, Q on $\mathbb{S}^2$, neither equal to N ,*

$$\theta_{Q \rightarrow P} - \theta_{P \rightarrow Q} - \pi \equiv \Delta\lambda - \tilde{E} \pmod{2\pi}. \quad (2.4)$$

Derivation. Assume first $\Delta\lambda > 0$ (the case $\Delta\lambda < 0$ follows by exchanging the roles of P and Q , and $\Delta\lambda = 0$ is immediate since then P, Q, N are collinear on a meridian and both sides of Eq. (2.4) vanish). The interior angle of the triangle at N is $C = \Delta\lambda$. The side PN lies along the meridian of P , which is exactly the reference direction (bearing 0) from which $\theta_{P \rightarrow Q}$ is measured; since Q is east of P , the interior angle at P is $A = \theta_{P \rightarrow Q} \in (0, \pi)$ directly. Symmetrically, the side QN is the reference direction at Q , but P lies *west* of Q , so the bearing $\theta_{Q \rightarrow P} \in (\pi, 2\pi)$ is a reflex angle relative to the triangle's interior, and the interior angle at Q is $B = 2\pi - \theta_{Q \rightarrow P}$. Girard's theorem gives $A + B + C = \pi + E$, i.e.

$$\theta_{P \rightarrow Q} + (2\pi - \theta_{Q \rightarrow P}) + \Delta\lambda = \pi + E,$$

which rearranges to $\theta_{Q \rightarrow P} - \theta_{P \rightarrow Q} - \pi = \Delta\lambda - E$; since $\Delta\lambda > 0$ here, $E = \tilde{E}$ and Eq. (2.4) follows. For $\Delta\lambda < 0$, the same argument with P, Q exchanged gives $\theta_{P \rightarrow Q} - \theta_{Q \rightarrow P} - \pi \equiv (-\Delta\lambda) - E \pmod{2\pi}$; negating and using $\tilde{E} = -E$ in this case recovers Eq. (2.4) again. $\square$

Remark 2.5. On the equator ($\varphi_P = \varphi_Q = 0$), the triangle P, Q, N has two right angles at P and Q (the equator meets every meridian at a right angle), so $E = \Delta\lambda$ identically and Eq. (2.4) gives a zero discrepancy: bearings along the equator behave exactly as in the planar case. This is the degenerate case in which the retroazimuthal problem trivialises, and it is consistent with $\tilde{E} = \Delta\lambda$ observed numerically below (Table 2.1, equatorial row).

Equation (2.4) has been checked numerically by two mutually independent computation paths (`code/bearing_asymmetry_spherical_excess.py`): the left-hand side from the `atan2` bearing formula (Eq. (2.2)) applied in both directions, the excess E on the right-hand side from *L'Huilier's formula*, which uses only the three side lengths of the triangle P, Q, N (colatitude of P , colatitude of Q , and the great-circle distance P – Q of Eq. (2.1)) and therefore shares no bearing computation with the left-hand side. Across 22 pairs formed from the 44 world cities already used elsewhere in the volume (Appendix A, taken consecutively in the order of that table) plus 6 deterministic geometric edge cases (equatorial, near-polar, straddling the equator, straddling the antimeridian, and sharing a meridian), the maximum absolute residual between the two sides of Eq. (2.4), reduced modulo 2π , is below 10^{-9} degrees (floating-point noise), i.e. the identity holds to the limits of double-precision arithmetic in every case tested. Full results in `data/bearing_asymmetry_spherical_excess_results.json`.

The near-polar row is instructive: at $\varphi = 89^\circ$ the two meridians subtend the same $\Delta\lambda = 90^\circ$ as the equatorial row, but the triangle P, Q, N is now extremely thin (both P, Q almost coincide with N), so $\tilde{E} \rightarrow 0$ and nearly the entire $\Delta\lambda$ survives as bearing-reversal discrepancy — the opposite regime from the equator, where curvature effects on bearing are, in this specific sense, maximal. This confirms quantitatively the qualitative observation of Remark 2.2: the discrepancy is a genuine curvature effect, it is not bounded by $\Delta\lambda$ itself, and it can be computed exactly, in closed form, once $\tilde{E}$ is available independently of the bearings whose difference it explains.

Table 2.1: Sample verification of Eq. (2.4) (selected rows; full table of 28 cases in the JSON output of `bearing_asymmetry_spherical_excess.py`).

$P \rightarrow Q$	$\Delta\lambda$	$\tilde{E}$	LHS	RHS	residual
Baghdad $\rightarrow$ Beijing	72.041°	25.088°	46.953°	46.953°	0°
Isfahan $\rightarrow$ Istanbul	-22.690°	-8.937°	-13.752°	-13.752°	0°
Equator, $\Delta\lambda = 90^\circ$	90°	90°	0°	0°	0°
Near-pole, $\varphi = 89^\circ$	90°	0.009°	89.991°	89.991°	0°
Sharing a meridian	0°	0°	0°	0°	0°

2.3 The direct and inverse geodetic problems

For completeness, recall the classical distinction, already used in the geodetic literature cited in the DGGS volume (this section revisits only the spherical, not the ellipsoidal, part of it):

- **Direct problem:** given P , an initial bearing θ and an angular distance c , find Q .
- **Inverse problem:** given P and Q , find c and $\theta_{P \rightarrow Q}$ (Eqs. (2.1) and (2.2)).

The cartographic retroazimuthal problem, formalised in the next chapter, requires solving the inverse problem for every point P on the map with respect to a single fixed point C , and encoding the result $\theta_{P \rightarrow C}$ as an angle directly readable on the map at every $f(P)$ — not only at one special point, as in the azimuthal case.

2.4 The «cotangent» form of the qibla formula

In the literature on the Islamic science of the qibla (Chapter 4) the bearing q toward Mecca from a locality of latitude φ at a longitude difference $\Delta\lambda$ from Mecca (of latitude φ_M) is often written in the form

$$\cot q = \frac{\sin \varphi \cos \Delta\lambda - \tan \varphi_M \cos \varphi}{\sin \Delta\lambda}. \quad (2.5)$$

Remark 2.6. This is algebraically almost the same expression that will reappear in Craig’s projection formula (Chapter 9, Eq. (9.1)): this is not a coincidence, but the direct trace of the historical kinship between the qibla problem and Craig’s cartographic construction.

The numerical computation carried out for this volume (code/`qibla_longitude_error_sensitivity.py`) shows that the bearing obtained by directly inverting Eq. (2.5) as $q = \text{atan2}(\sin \Delta\lambda, \text{numerator})$ differs systematically from the bearing $\theta_{P \rightarrow C}$ of Eq. (2.2) by a value that, in the five test cases (Italian cities, Table 2.2), turns out to be exactly the supplement: $q + \theta_{P \rightarrow C} = 180^\circ$ to within numerical precision (the two agree with the expected value $180^\circ - \theta$ to better than 10^{-3} degrees). This is consistent with a different convention for measuring the angle q (e.g. from local south rather than north, or a different branch of the arc-cotangent) in the secondary source from which Eq. (2.5) was transcribed, but it has not been possible to verify which exact convention the primary source (a historical *zij*) uses without direct access to the Arabic text or a critical edition. The equivalence of the two formulas is therefore reported

as verified *up to this convention ambiguity*, explicitly flagged, not as an absolute identity.

Table 2.2: Comparison of the two forms of the qibla bearing (Belluno, Rome, Venice, Palermo, Milan; Mecca at 21.3891°N, 39.8579°E). Values computed by `qibla_longitude_error_sensitivity.py`.

City	$\theta_{P \rightarrow C}$ (atan2, Eq. (2.2))	q (cotangent, Eq. (2.5))	$\theta + q$
Belluno	128.372°	51.628°	180.000°
Rome	123.277°	56.723°	180.000°
Venice	127.667°	52.333°	180.000°
Palermo	118.700°	61.300°	180.000°
Milan	123.563°	56.437°	180.000°

The rest of the volume uses exclusively the atan2 form of Eq. (2.2), mathematically unambiguous and free of quadrant ambiguity, reserving the cotangent form to the historiographic comparison of Section 2.4.

Chapter 3

Formal Definition and a First Non-Coincidence Result

3.1 Formal definition

We formalise what was anticipated in Chapter 1.

Definition 3.1 (Cartographic north). Given a differentiable projection f and a point P at which f is locally invertible, **cartographic north** at $f(P)$ is the direction of the image, under the differential df_P , of the unit tangent vector pointing toward the North Pole along P 's meridian.

Definition 3.2 (Retroazimuthal projection, explicit form). Let $C = (\varphi_M, \lambda_M)$ be a fixed point. A projection $f : \mathbb{S}^2 \setminus D \rightarrow \mathbb{R}^2$ (where D is the, possibly non-empty, set of singular points, typically C and its antipode) is **retroazimuthal with respect to C** if for every $P \notin D$

$$\text{atan2}(x(C) - x(P), y(C) - y(P)) = \theta_{P \rightarrow C} \pmod{2\pi}, \quad (3.1)$$

where $\theta_{P \rightarrow C}$ is the initial great-circle bearing defined in Eq. (2.2) and the $+y$ axis of the image plane is aligned, by construction of the projection, with cartographic north at every point (a condition automatically satisfied by all three constructions studied in this volume, all having either straight or locally y -aligned meridians).

This is the operative form used in the checks of Chapters 9 and 10 (function `map_bearing_to_origin` in `code/verify_retroazimuthal_property.py`): the angle, measured from north, of the segment joining the projected point to the projected fixed point, must equal the true spherical bearing.

3.2 A provable non-coincidence result

Before tackling (in Chapter 13) the deeper and more delicate question — whether a global retroazimuthal projection can be conformal — it is worth isolating a more elementary fact, often left implicit in popular accounts: azimuthality and retroazimuthality with respect to the same centre C are, in general, incompatible properties for the same projection.

Proposition 3.3 (Generic non-coincidence). *Let f be a projection that is azimuthal with respect to C (i.e. such that for every P the angle of segment $f(C)f(P)$ from north at $f(C)$ equals $\theta_{C \rightarrow P}$). If f is also retroazimuthal with respect to C in the sense of Definition 3.2, then for every P such that $f(C), f(P)$ are not aligned with a degenerate axis,*

$$\theta_{C \rightarrow P} = \theta_{P \rightarrow C} + \pi \pmod{2\pi}. \quad (3.2)$$

But Eq. (3.2) is generally false on the sphere (Remark 2.2): there exist pairs (C, P) for which $\theta_{C \rightarrow P} - \theta_{P \rightarrow C} - \pi \not\equiv 0 \pmod{2\pi}$, except in the degenerate case in which C, P , and the two poles are related

by a symmetry that annihilates the relevant spherical excess (e.g. C and P on the equator, or C and P on the same meridian). Hence no projection can be simultaneously azimuthal and retroazimuthal with respect to the same centre C on a domain containing non-degenerate pairs.

Proof. If f is azimuthal at C , the angle of $f(C)f(P)$ from north at $f(C)$ is $\theta_{C \rightarrow P}$ by definition. Since $f(C)f(P)$ is the same segment as $f(P)f(C)$ (trivially true, it is the same segment traversed in the two directions), the angle it forms as seen from $f(P)$ is, in the Euclidean plane in which the image lives, exactly the angle seen from $f(C)$ plus π (angles of a segment at its two endpoints, measured from the same fixed direction $+y$ in the image plane, always differ by exactly π : this is an elementary property of the Euclidean geometry of the image plane, independent of f). If f is also retroazimuthal at C , this same angle must equal $\theta_{P \rightarrow C}$. Hence $\theta_{C \rightarrow P} = \theta_{P \rightarrow C} + \pi$, which is Eq. (3.2). The generic falsity of the latter follows from the non-symmetry of spherical bearing (Remark 2.2), verifiable numerically for any non-degenerate triple of points by choosing C, P neither both on the equator nor on the same meridian. $\square$

Remark 3.4. The proof uses a purely Euclidean fact about the *image plane* (angles at the endpoints of a segment differ by π), applied to the spherical fact of bearing non-symmetry. It is therefore not specific to any of the three projections studied: it holds for any hypothetical projection, which is precisely the point. It is also the reason why, as will be seen, the constructions of Craig and Hammer make no attempt to be azimuthal with respect to Mecca at all: they give up that property entirely in order to obtain the retroazimuthal one.

3.3 Toward conformality: what this chapter does NOT prove

Methodological Note

Chapter 13 will discuss the claim, found in the standard cartographic literature (repeated by Snyder 1993 and by the cartographic encyclopaedic entry on the *Littrow projection*, but traced in Chapter 13 and Appendix B.3 to Maurer 1919 as its actual origin, with Snyder reporting it only at second hand), that the Littrow projection is the «only conformal retroazimuthal projection». As transmitted by the secondary sources consulted for this volume, this claim is not accompanied by a proof reproduced in full in the text consulted online. This volume therefore does not claim to have reproduced the original proof (found in Maurer 1919, a German-language paper not consulted directly for this volume; obtaining Snyder 1993 in full would not by itself supply it, since Snyder is not the original source). What Chapter 13 offers is (i) a heuristic argument based on the rigidity of conformal maps of the sphere minus a point, and (ii) a specific numerical check — not a general proof — showing that attempting to generalise Littrow to an arbitrary centre by rotating the sphere *fails* explicitly for the centre corresponding to Mecca (Chapter 8). The combination of the two arguments is presented as evidence in favour of the established claim, not as a new original proof of it.

Part II

The Historical Case: The Qibla

Chapter 4

The Qibla Problem in Medieval Islamic Astronomy

Methodological Note

This chapter draws on the secondary literature synthesised by David A. King, the historian of Islamic astronomy whose work (notably *Astronomy in the Service of Islam*, 1993, and *World-Maps for Finding the Direction and Distance of Mecca*, 1999) is the standard reference on the subject. For this volume, abstracts, summaries, and related articles available online were consulted (King 1996, 2020; Kennedy & Kennedy 1987 via Mercier, *e-Perimetron* 15(2)), not the full editions of the cited volumes themselves. The claims reported here are therefore limited to what is explicitly attested in these consulted secondary sources; finer technical details (exact formulas of individual astronomers, original numerical tables) are flagged with a *todobox* where they could not be directly verified from the available sources.

4.1 The problem and its ritual importance

In Islam, the sacred direction — the *qibla* — is the direction toward the Kaaba in Mecca; Muslims face this direction in ritual prayer and various other acts of worship. From the eighth century onward, Muslim astronomers addressed the determination of the qibla as a problem of mathematical geography, producing trigonometric and geometric solutions of considerable sophistication.

The problem has a two-fold structure, clearly distinguished in King's historiography:

1. a tradition of **sacred geography** (or «folk astronomy»), in which the world around the Kaaba is divided into sectors, each associated with a horizon astronomical phenomenon (the rising or setting of a star, the prevailing wind) that defines its qibla direction in an approximate but immediately usable way;
2. a tradition of **mathematical geography**, in which the qibla is calculated from geographic coordinates (longitude and latitude, in the Ptolemaic-derived geographers' tradition) by exact or approximate trigonometric or geometric procedures.

This dichotomy — not a simple linear evolution from the «primitive» to the «sophisticated», but the persistence of two parallel traditions into modern times — partly explains why the orientation of historical mosques does not always match the value that exact spherical trigonometry, applied to modern coordinates, would prescribe: architects and local religious jurists often followed the sacred-geography tradition, cruder but sanctioned by custom, while court astronomers developed increasingly refined mathematical solutions in parallel.

4.2 Essential chronology

- **From the eighth century:** Muslim astronomers begin tackling the determination of the qibla as a problem of mathematical geography, starting from the coordinates of Mecca and of the locality in question.
- **From the ninth century:** approximate methods for computing the qibla, accurate enough for everyday practical use and simpler to construct than the exact methods, become widely known; most *zijs* (medieval astronomical handbooks with tables) contain a chapter devoted to the mathematical determination of the qibla, as well as geographic tables giving the longitude and latitude of numerous localities.
- An approximate method attributed to **al-Battani** (the Syrian astronomer active at Raqqa, who wrote around 297 AH / 910 CE) was widely used and remained popular until the nineteenth century.
- Astronomers also developed exact trigonometric and geometric solutions of considerable sophistication, eventually compiling tables showing the qibla direction for every degree of latitude and longitude.

Key Result

The explicit formula has since been retrieved, reported by King (2019) in the course of his study of the Cordoba mosque's orientation. Given the latitude difference $a - b$ between locality and Mecca and the longitude difference c between them (both, in al-Battani's method, simply subtracted as plane quantities, with no spherical correction), the qibla angle q , measured from the local meridian, is

$$q = \arctan\left(\frac{\sin c}{\sin(a - b)}\right). \quad (4.1)$$

King also reports the method's geometric construction, consistent with the algebraic form above: lay out the cardinal directions on a circle around a centre O ; mark a point G on the arc between east and south such that the arc EG equals the latitude difference $a - b$; mark a point K on the arc between south and east such that the arc SK equals the longitude difference c ; draw a line through G parallel to the east–west diameter and a line through K parallel to the north–south diameter, meeting at a point Z ; the line from O through Z gives the qibla direction, read off as the arc from south.

Methodological Note

The formula is a planar (flat-Earth) approximation: it treats latitude and longitude differences as if they were Cartesian offsets on a plane tangent at the locality, with no spherical trigonometry beyond the single ratio of sines. King's own assessment, and that of the sources cited earlier in this chapter more generally, is that this is precisely why the method remained popular for its simplicity while being accurate only for localities reasonably close to Mecca, degrading steadily with distance — consistent with what a planar approximation of a spherical-trigonometric quantity would be expected to do. [Chapter 6](#) does not use this formula: its sensitivity analysis was designed, and is retained here, around the exact spherical formula subjected to a documented input-coordinate error ([Section 4.3](#)), which isolates a single, independently quantified source of error rather than compounding it with the error — now known exactly, but not analysed here — intrinsic to al-Battani's own approximation.

4.3 The systematic Ptolemaic-Islamic geographic error

Independently of the accuracy of the qibla *calculation method* (exact or approximate), any calculation — ancient or modern — depends on the quality of the starting geographic coordinates. Here the history of mathematical cartography meets directly the history of qibla astronomy: the geographic tradition inherited from Ptolemy (*Geographia*, second century CE) contained a systematic error, well documented and quantified in the literature, in the longitudinal extent attributed to the Mediterranean Sea.

Key Result

Ptolemy overestimated the longitudinal extent of the Mediterranean (from the Canary Islands, his reference meridian, to its eastern shores) at about 63° , against a correct value of about 50° : an error of about 13° , arising predominantly from his underestimate of the Earth's circumference (his adoption of Posidonius's estimate, which understated the length of a meridian degree by roughly one sixth). Al-Khwarizmi, the best-known geographer at al-Ma'mun's court in Baghdad (*Kitab surat al-Ard*, completed in 833), corrected this estimate to about 50° , a value close to the correct one. The residual error, by construction, grows progressively as one moves further east from the western reference meridian (the Canary Islands).

This historical fact — independent of any particular qibla calculation method, and quantified in the literature independently of it — provides the basis for the sensitivity analysis in the next chapter: any qibla calculation performed by a medieval astronomer using geographic coordinates derived (even indirectly) from the uncorrected Ptolemaic tradition would have inherited a relative longitude error depending on the locality's position relative to the adopted reference meridian.

Methodological Note

It should be stated clearly what [Chapter 6](#) does *not* do: it does not attribute a specific, documented longitude error to a particular geographic source for Italy (Belluno, Rome, etc. do not appear, as far as verified, in the medieval Islamic geographic tables ordinarily studied, which focus on the Islamic world and the eastern/southern Mediterranean). The analysis treats the longitude shift as an explicit sensitivity parameter, with sign left unspecified a priori, within the historically plausible range suggested by the magnitude of the documented overall Ptolemaic-Islamic error — not as a reconstruction of a specific historical value for a qibla actually calculated by a particular astronomer for these localities.

4.4 From mathematical geography to cartography: qibla maps

King documents that, beyond numerical calculations and tables, the Islamic tradition also produced actual *maps* for the graphical determination of the qibla, including at least two seventeenth-century Safavid maps based on mathematical grids whose origin, according to King, traces back to tenth-century work in Baghdad and eleventh-century work in Isfahan. A more surprising claim, reported by Waldo Tobler (2002) on the basis of a recent discovery from Iran, suggests that Muslims may have anticipated, by several centuries, a construction akin to the Western retroazimuthal one of Craig/Hammer.

Key Result

The «recent discovery from Iran» behind Tobler’s remark can now be identified: two antique Iranian instruments, engraved with cartographic grids for finding the direction and distance to Mecca, studied by King (1997, *Imago Mundi* 49) and discussed again by Dekker (2000, *The Cartographical Journal* 37). The two scholars disagree on how to read the grids. King’s account (as summarised by Tobler) treats them as evidence of an indigenous Islamic retroazimuthal construction anticipating Craig and Hammer by several centuries. Dekker instead argues the engraved curves are well explained by the *Mecca-centred retroazimuthal orthographic* projection described by J. E. Jackson in 1968 — latitude circles reducing to ellipses with major axis parallel to the equator, longitude lines reducing to a family of non-equidistant straight lines — and, according to a contemporary summary of the exchange, does not accept either that the latitude curves are genuinely circular arcs (as King’s reading requires) or that the inspiration for the grids is Islamic rather than a later, Western-derived construction retrospectively applied to Iranian instruments.

Methodological Note

Two things follow from identifying the primary dispute rather than only Tobler’s one-paragraph abstract. First, the claim is more precisely locatable than «a recent discovery from Iran»: it concerns specific instruments, specific publications, and a named counter-reading, not a vague report. Second, and for that very reason, it is *less*, not more, safe to repeat as an established fact: what is on record is an active scholarly disagreement about attribution and technical description, not a settled result. The retroazimuthal orthographic construction Dekker proposes is also, in any case, of a different geometric type (azimuthal-orthographic) from the two constructions studied at length in this volume, Craig (cylindrical) and Hammer (a distinct azimuthal construction) — so even on King’s own reading, the claimed Islamic precursor would not be a precursor of *this volume’s* Craig or Hammer constructions specifically, only of the broader retroazimuthal idea. Consulting King (1997) and Dekker (2000) directly, rather than the summaries used here, remains a legitimate further step, left open — taken up, as far as it can be without full access to either paper, in [Chapter 5](#).

It was precisely this background — centuries of mathematical tradition on the qibla, combined with the full maturation of Western mathematical cartography around the turn of the twentieth century — that made the explicitly cartographic formulation of the problem by Craig and Hammer, in 1909–1910, a natural next step; this is the subject of Part III of this volume.

Chapter 5

Consulting King (1997) and Dekker (2000) Directly

Methodological Note

Chapter 4 identified the primary dispute behind Tobler's (2002) one-paragraph remark on a possible Islamic precursor of the retroazimuthal idea, but left «consulting King (1997) and Dekker (2000) directly, rather than the summaries used here,» as an explicitly open further step. This chapter takes that step, as far as it can be taken without full access to either paper: a renewed search located King's (1997) own published abstract (via the journal's own platform, Taylor & Francis Online) and, for Dekker (2000), excerpts that include not only the abstract but the paper's own concluding paragraph and two further technical passages (via a scholarly aggregator's cached preview). This is still short of the full text of either paper — so still flagged, honestly, as partial — but it is a genuine step closer to the primary sources than Chapter 4 had reached, which relied for Dekker (2000) on King's own gloss of it in his annotated publication list, not on Dekker's own words.

5.1 King (1997), consulted directly: what the abstract actually claims

Key Result

King's (1997) own abstract, now consulted directly rather than via Tobler's (2002) paraphrase or Chapter 4's own secondary account, is more hedged than either. Two Islamic brass instruments, each bearing a world map for finding the direction and distance to Mecca, are described as having come to light only since 1989; both appear to date from Safavid Iran, plausibly the late seventeenth century; each is a copy of a different earlier map of the same kind, and both of those are said to derive from a *more elaborate prototype* that is not itself among the two surviving instruments. King's own abstract states that these two instruments are the only known surviving Islamic world maps with localities properly marked on a competently drawn coordinate grid — and, on the question of antiquity, uses the word «likely», not a stronger term: that serious activity in mathematical cartography, including detailed world maps, is known to have started in Baghdad in the ninth century, and that it is *likely* the sophisticated cartographic grid technique of the two surviving instruments was first developed some centuries before the instruments themselves were made.

Methodological Note

This confirms, from King's own wording rather than only from Chapter 4's secondary account of it, that the strong form of the precursor claim (an Islamic construction antedating Craig and Hammer «by several centuries», as Chapter 4 put it, following Tobler's

paraphrase) rests on an inference King himself calls likely, not on a documented chain of surviving intermediate sources connecting the ninth-century Baghdad tradition to the two seventeenth-century instruments. Nothing here overturns Chapter 4's caution; if anything, seeing King's own hedge («likely») in place of Tobler's flatter summary sharpens it.

5.2 Dekker (2000), consulted directly: a more precise, and more measured, counter-reading

Chapter 4 reported Dekker's position, via King's own annotated bibliography, as refusing to accept either that the engraved curves are circular arcs or that the grids are of Islamic inspiration. Consulting excerpts of Dekker's own paper directly gives a more precise, and in one respect more measured, picture.

Key Result

The technical claim. Dekker's construction, following J. E. Jackson's 1968 Mecca-centred retro-azimuthal orthographic projection, predicts that the lines of constant latitude on such a grid reduce to ellipses with major axis parallel to the equator, and the lines of constant longitude to a family of non-equidistant straight lines parallel to the north-south direction; cities are placed at a distance from the centre proportional to the sine of the great-circle distance to Mecca, which is exactly the condition that keeps the direction to Mecca correct at every point — the retroazimuthal property itself, constructed by an entirely different route from Craig's or Hammer's. Dekker reports that the curves actually engraved on the instruments conform to this construction and can be reproduced from it without great difficulty.

Key Result

The circle-versus-ellipse point, more precisely. King's own account (1999, pp. 242–254, as cited in Dekker's text) reads the engraved latitude curves as circular arcs. Dekker's own reasoning, consulted directly rather than via King's gloss, is more measured than a flat rejection: her own construction predicts ellipses, not circles, but she notes that circles are easier to engrave than ellipses and that, over the limited area close to the map's centre where the surviving instruments' grids are concentrated, an ellipse and a circle are difficult to tell apart. The disagreement is therefore a genuine technical one — her derivation predicts ellipses, King reads circles — but Dekker's own text frames it as a fine distinction possibly lost to workmanship, not (as King's bibliographic gloss might suggest to a reader who has not seen Dekker's own text) an outright refusal to countenance King's reading.

Key Result

The origin question, more precisely. This is where consulting Dekker directly changes the picture furthest from Chapter 4's secondary account. King's own annotated bibliography glosses Dekker's conclusion as refusing to accept an Islamic inspiration for the grids outright. Dekker's own concluding paragraph, as excerpted, is narrower and more agnostic than that gloss suggests: it states that the instruments' workmanship is not accurate enough to distinguish her reading from King's by studying the instruments alone, and that her own geometric analysis undercuts specifically the claim of a *direct* line of descent from

the ninth-century Baghdad mathematical-cartography tradition King invokes — not the broader, weaker claim that the grids are Islamic in some sense. Her own stated conclusion is that it remains, in her own assessment, entirely unknown where and when the idea behind the Iranian grid was first conceived, and that the question of its historical background is still open — a narrower and more agnostic claim than «the inspiration is not Islamic».

The dispute, reconsulted at the primary-source level (as far as available), is therefore not King's Islamic-precursor claim versus Dekker's rejection of it, but King's specific claim of a documented several-centuries-old continuous Islamic tradition versus Dekker's narrower objection that the ninth-century Baghdad connection specifically does not follow from the geometry, combined with Dekker's own explicit agnosticism about the deeper origin question. This is a real refinement, not a semantic one: [Chapter 4](#)'s disclaimerbox already flagged the exchange as «an active scholarly disagreement about attribution and technical description, not a settled result», and that remains true; what changes is that the disagreement, read in Dekker's own words rather than in King's summary of them, turns out to leave more room open than a simple «King says yes, Dekker says no» would suggest. This is exactly the kind of distortion that can enter a chain of citation — not fabrication, but a reviewer's own gloss of a rival's position, repeated by a third party ([Chapter 4](#)) without the underlying text in hand — and is worth recording as a small methodological lesson for the rest of this volume's own citation practice, most of which likewise still rests on abstracts and excerpts rather than full texts ([Appendix B.3](#)).

5.3 One further, previously unrecorded detail: Schoy's variant

Methodological Note

Consulting Dekker's text directly also surfaced one concrete fact absent from [Chapter 4](#) and not previously recorded anywhere in this volume: a third name, alongside Craig and Hammer, in the retroazimuthal genealogy Dekker discusses.

Key Result

Dekker's text reports that a later corollary due to Carl Schoy leads to a distinct, «cylindrical» azimuthal map construction, in which the parallels bend *away from* the equator — the opposite sense to the standard retroazimuthal parallels of Craig, Hammer, and the Jackson-type construction discussed above, which bend toward it ([Section 13.8](#)). Dekker's excerpted text describes this Schoy variant as illustrated there for the first time, which, if accurate, would mean no published figure of it existed before Dekker's own 2000 paper.

To Complete / Verify

Concretely open after this chapter: (i) the full texts of both King (1997) and Dekker (2000) remain unobtained; everything above is reconstructed from abstracts and excerpts located via search, not from a complete reading of either paper, and should be revised if and when the full texts become available; (ii) Schoy's own construction and its date are not otherwise documented in this volume and have not been independently verified beyond Dekker's own passing mention; whether it is genuinely a nineteenth-, twentieth-century,

or otherwise-dated work, and whether «cylindrical» here means anything related to this volume's own use of the term for Craig ([Chapter 9](#)), is left entirely open; (iii) King's 1999 monograph itself (pp. 242–254 in particular) has not been consulted directly at any point in this volume, only via Dekker's citation of it and via the book's own index terms; a direct reading of those pages would settle, rather than only narrow, the circle-versus-ellipse disagreement of [Section 5.2](#).

Chapter 6

Sensitivity of the Qibla Bearing to Historical Geographic Errors

6.1 Setup of the analysis

This chapter applies the exact bearing formula (Eq. (2.2)) to a small set of Italian localities — not to reconstruct a qibla actually calculated historically (no medieval Islamic source, as far as verified, explicitly treats these localities), but to quantify, with a controlled and transparent experiment, how a relative longitude error of the order documented in the Ptolemaic-Islamic tradition (Section 4.3) translates into a bearing error. The goal is to illustrate, with real, reproducible numbers, the order of magnitude of the effect, not to attribute a specific historical value to any qibla actually calculated in the past.

Methodological Note

Choice of localities: Belluno, Rome, Venice, Palermo, Milan. Belluno is included for continuity with the author’s toponymic and local interests documented elsewhere in his output; the other four cover a sufficiently wide range of Italian latitudes and longitudes to show how the effect varies with position. The coordinates used are nominal current WGS84 values (not historical); Mecca is fixed at 21.3891°N, 39.8579°E, the same reference coordinates used in Chapters 9 and 10 for the verification of the projections, so that the two chapters share the same fixed point.

6.2 Results

Table 6.1 reports, for each locality, the bearing computed with correct modern coordinates (column «baseline») and the bearing error resulting from a systematic shift $\pm\Delta\lambda$ applied to the locality’s longitude alone, for $\Delta\lambda \in \{2^\circ, 5^\circ, 8^\circ, 13^\circ\}$ (the limiting value 13° corresponds to the entire residual Ptolemaic overestimate relative to al-Khwarizmi’s correction, $63^\circ - 50^\circ$, Section 4.3, used here as an illustrative limiting case). All values were computed by `code/qibla_longitude_error_sensitivity.py` and are reported without further rounding beyond that already present in the `qibla_sensitivity_results.json` file generated by the script.

Key Result

For all five localities, a relative longitude error of 5° — a modest fraction, less than half, of the documented residual Ptolemaic error — already produces a qibla bearing error between 6° and 7.3° : an error clearly perceptible on any map or compass, and sufficient to significantly deviate a building’s orientation. At the far end of the tested range (13° , the entire residual Ptolemaic error), the bearing error exceeds 19° – 22° in every case. The non-

Table 6.1: Qibla bearing error (degrees) induced by a systematic longitude shift $\Delta\lambda$, for the five test localities. Column «base»: exact bearing with modern coordinates. Following columns: |error| for an eastward shift ($+\Delta\lambda$) and a westward shift ($-\Delta\lambda$).

City	base θ ($^\circ$)	$\Delta\lambda = 2^\circ$		$\Delta\lambda = 5^\circ$		$\Delta\lambda = 8^\circ$		$\Delta\lambda = 13^\circ$	
		+	−	+	−	+	−	+	−
Belluno	128.372	2.813	2.680	7.298	6.470	12.123	10.010	20.958	15.428
Rome	123.277	2.792	2.626	7.327	6.285	12.327	9.652	21.818	14.721
Venice	127.667	2.819	2.680	7.326	6.462	12.193	9.985	21.150	15.362
Palermo	118.700	2.744	2.544	7.292	6.037	12.447	9.200	22.670	13.891
Milan	123.563	2.607	2.483	6.770	5.997	11.262	9.283	19.555	14.329

linearity is evident and expected: Eq. (2.2) is not linear in $\Delta\lambda$, and the asymmetry between eastward and westward shifts (the + and − columns are not identical) reflects each locality's specific position relative to the great circle connecting it to Mecca.

6.3 Cautious interpretation

Methodological Note

This experiment quantifies the sensitivity of the *exact* formula to an error *in the input* (the coordinates), not the error of a historical *approximate* method applied to correct coordinates: these are two conceptually distinct sources of error, and Chapter 4 has already noted that the second cannot be reconstructed for this volume without access to primary sources (*todo* of Chapter 4). The order of magnitude computed here (a few degrees per 2–3 degrees of longitude error) is nonetheless instructive: it shows that, historically, even an astronomer in possession of the *exact* trigonometric formula would still have produced appreciably wrong qibla bearings if the starting geographic coordinates — the real bottleneck, more than the calculation method — had been those, systematically distorted, of the not-yet-corrected Ptolemaic tradition. This is consistent with King's own observation that the history of the qibla is as much a history of spherical trigonometry as it is a history of mathematical geography and of the quality of its coordinate tables.

Chapter 7

Al-Battani's Approximate Method: A First-Order Derivation and Quantitative Error Analysis

Chapter 4 reports al-Battani's approximate qibla formula as retrieved by King (2019),

$$q = \arctan\left(\frac{\sin c}{\sin(a - b)}\right), \quad (7.1)$$

attributing to it a «flat-Earth» character without working out precisely what that means or how large its error is. This short chapter takes up that open point, flagged in Chapter 15 as «not verified in this volume, reported only with caution», and closes it as far as the volume's own resources allow. Three things are done, in order: Equation (7.1) is related, by an original first-order expansion of the exact bearing formula (Eq. (2.2)) around Mecca, to a precise local tangent-plane approximation; a genuine ambiguity in King's report — the subtraction order of « $a - b$ » — is resolved by requiring consistency with that expansion, not assumed; and the resulting formula is evaluated, exactly as resolved, against the exact bearing on the volume's own forty-four-city table (Appendix A), giving the error its literature description («accurate near Mecca, degrading with distance») had never been assigned a number.

Methodological Note

Everything in this chapter traces to `code/al_battani_approximation_analysis.py` and the JSON results it produces (`data/al_battani_approximation_analysis_results.json`), reusing verbatim the same Mecca reference point and the same forty-four-city coordinate table as Appendix A and Chapter 6. No new coordinates are introduced. Al-Battani's original treatise was not consulted for this volume, only King's (2019) secondary report of the formula (already so flagged in Chapter 4); the sign-convention argument of Section 7.2 is this volume's own resolution of an ambiguity in that report, checked for internal geometric consistency, not a claim about what al-Battani himself intended or what his lost diagram actually showed.

7.1 A local tangent-plane expansion of the exact formula

Write φ_C, λ_C for Mecca's coordinates and $\varphi = \varphi_C - \varepsilon, \lambda = \lambda_C - c$ for a locality displaced from Mecca by a small latitude offset ε and longitude offset c (both in radians; the sign choice matches Section 7.2 below). Substituting into the exact bearing components of Eq. (2.2),

$$x = \cos \varphi \sin \varphi_C - \sin \varphi \cos \varphi_C \cos(\lambda_C - \lambda), \quad y = \sin(\lambda_C - \lambda) \cos \varphi_C,$$

and expanding $\cos \varphi = \cos(\varphi_C - \varepsilon), \sin \varphi = \sin(\varphi_C - \varepsilon)$ to first order in ε , and $\lambda_C - \lambda = c$ to first order in c (no expansion needed there, it already appears alone), gives

$$x \approx \varepsilon (\sin^2 \varphi_C + \cos^2 \varphi_C) = \varepsilon, \quad y \approx c \cos \varphi_C,$$

to leading order (the algebra is the same double-angle collapse used in [Proposition 2.4](#) for the polar-triangle excess). The bearing $\theta = \text{atan2}(y, x)$ near Mecca is therefore approximated, to first order, by

$$\theta_{\text{tangent}} \approx \text{atan2}(c \cos \varphi_C, \varepsilon). \quad (7.2)$$

Key Result

This is an ordinary local-tangent-plane (equirectangular, small-angle) approximation: north displacement ε read directly in radians, east displacement c scaled by $\cos \varphi_C$ to account for the convergence of meridians at Mecca's latitude. The scaling factor is not optional: dropping it corresponds to treating one degree of longitude as the same physical length as one degree of latitude, which is false away from the equator by exactly the factor $\cos \varphi_C = \cos(21.3891^\circ) = 0.931125$ recovered by the script.

Equation (7.2) was checked, independently of the by-hand algebra above, by direct finite-difference sampling of the exact components x, y on a shrinking grid of offsets around Mecca ($\pm 0.05^\circ$ down to $\pm 0.0016^\circ$, halved five times, 25×25 points per grid). [Table 7.1](#) reports the maximum relative deviation of the linear model from the exact components at each scale.

Max offset ($^\circ$)	max rel. error, x	max rel. error, y
0.05000	1.49×10^{-4}	1.18×10^{-7}
0.02500	7.42×10^{-5}	2.95×10^{-8}
0.01250	3.71×10^{-5}	7.39×10^{-9}
0.00625	1.85×10^{-5}	1.85×10^{-9}
0.00313	9.26×10^{-6}	4.62×10^{-10}
0.00156	4.63×10^{-6}	1.15×10^{-10}

Table 7.1: Finite-difference check of [Eq. \(7.2\)](#): the relative error in both components roughly halves each time the offset halves, confirming first-order (linear) convergence, as expected of a Taylor truncation, rather than an accidental agreement at one scale.

7.2 Resolving the subtraction-order ambiguity

[Chapter 4](#) reports « $a - b$ » only as «the latitude difference $a - b$ between locality and Mecca», without stating which of the two is a and which is b ; King's own secondary account, consulted only as excerpted online, does not disambiguate this further, and the geometric construction quoted alongside it (points G, K, Z on a compass circle) is described with an apparent internal repetition («on the arc between east and south» and «on the arc between south and east» for two different points) that this volume cannot resolve from the source as consulted.

To Complete / Verify

The subtraction order of $a - b$ in [Eq. \(7.1\)](#) is genuinely ambiguous in the source as consulted here. Direct access to King (2019) in full, or to al-Battani's own *Zij*, would settle it without recourse to the argument below.

Lacking that access, the ambiguity is resolved here by an internal consistency requirement rather than left unresolved: of the two possible readings, $a - b = \varphi_{\text{locality}} - \varphi_C$ or $a - b = \varphi_C - \varphi_{\text{locality}}$, only one can agree in sign with the independently derived tangent-plane expansion ([Eq. \(7.2\)](#)) for localities close to Mecca, where any correct approximate method *must* agree

closely with the exact formula by construction. The script tested both directly: taking $a - b = \varphi_{\text{locality}} - \varphi_C$ (locality minus Mecca) gives a formula that, evaluated on the volume's nearest cities to Mecca — Riyadh at 789 km, Sana'a at 810 km — misses the exact bearing by 52.6° and 110.6° respectively: an approximation that fails grossly at zero range cannot be the intended one. Taking instead $a - b = \varphi_C - \varphi_{\text{locality}}$ (Mecca minus locality, i.e. $a - b = \varepsilon$ in the notation of Section 7.1) reduces the same two errors to 0.52° and -2.09° . The second reading is adopted from here on; by the same consistency argument, c is read as $\lambda_C - \lambda_{\text{locality}}$ (Mecca minus locality), matching the sign of $\Delta\lambda$ already used throughout the volume's exact formula.

Methodological Note

This resolves an ambiguity in how *this volume reads* King's (2019) report into a full 0° – 360° bearing formula comparable to the rest of the volume; it is not a claim about which convention al-Battani's own text used, nor a correction of King. The geometric construction quoted in Chapter 4 (points G, K, Z) may resolve the same ambiguity unambiguously if read correctly from a full edition, but that reading was not achieved here.

With the ambiguity resolved this way, al-Battani's formula, cast as a full bearing in the same atan2 convention as the rest of the volume, is

$$\theta_{\text{Battani}} = \text{atan2}(\sin c, \sin \varepsilon) \bmod 360^\circ, \quad \varepsilon = \varphi_C - \varphi, \quad c = \lambda_C - \lambda. \quad (7.3)$$

Comparing Eq. (7.3) with Eq. (7.2) term by term, $\sin \varepsilon \approx \varepsilon$ recovers the tangent-plane's north component exactly to first order, but $\sin c$ carries no $\cos \varphi_C$ factor: al-Battani's formula, as resolved here, is the correct first-order approximation in the north–south direction but omits the meridian-convergence correction in the east–west direction. The size of that omission, $1/\cos \varphi_C - 1 = 0.0740$, sets an expected systematic bias of a few per cent on the longitude term alone, distinct from — and, as Section 7.3 shows, much smaller than — the error introduced simply by using a linear approximation at non-small distances.

7.3 Quantitative error against the exact formula

Equation (7.3) was evaluated, together with the exact bearing (Eq. (2.2)) and the idealised tangent-plane approximation (Eq. (7.2)), on all forty-four cities of Appendix A, ranging from Riyadh (789 km from Mecca) to Los Angeles (13429 km). Table 7.2 reports a representative subset spanning that full range; the complete forty-four-row table is in the JSON results file.

Table 7.2: Al-Battani's approximate bearing (Eq. (7.3)) against the exact bearing, selected cities by increasing distance from Mecca.

City	Dist. (km)	Exact ($^\circ$)	Al-Battani ($^\circ$)	Error ($^\circ$)
Riyadh, Saudi Arabia	789	243.45	243.96	+0.52
Sana'a, Yemen	810	326.32	324.23	−2.09
Baghdad, Iraq	1398	199.64	200.82	+1.18
Muscat, Oman	1920	266.33	263.13	−3.20
Istanbul, Turkey	2410	151.58	150.66	−0.92
Rome, Italy	3424	123.28	127.32	+4.05
Belluno, Italy	3718	128.37	132.06	+3.69
Delhi, India	3833	266.52	258.29	−8.23
Paris, France	4501	119.16	127.15	+7.99

City	Dist. (km)	Exact (°)	Al-Battani (°)	Error (°)
London, United Kingdom	4799	118.98	127.99	+9.00
Dhaka, Bangladesh	5170	277.51	266.87	−10.64
Beijing, China	7385	278.84	251.92	−26.92
Tokyo, Japan	9471	292.95	255.94	−37.01
New York, USA	10311	58.48	109.89	+51.41
Sydney, Australia	13232	277.48	311.42	+33.94
Los Angeles, USA	13429	23.84	120.45	+96.61

Key Result

Restricted to the nine cities within 2000 km of Mecca — the range of the historical zij tradition’s core geographic horizon, covering Arabia, the Levant, Iraq, and Iran — the mean absolute error of Eq. (7.3) is 1.28° , well inside the tolerance the historical sources describe the method as having for localities close to Mecca (Chapter 4). Over all forty-four cities, spanning every inhabited continent up to 13429 km, the mean absolute error rises to 13.6° , and the correlation between distance and absolute error is strong and positive ($r = 0.81$): the qualitative literature description («degrading with distance») is confirmed quantitatively, not just asserted. Beyond 2000 km the mean absolute error is 16.8° , more than an order of magnitude worse than the near-Mecca figure; the very largest errors, exceeding 90° , occur only for the Americas and Australia, at antipodal-scale distances the ninth-century mathematical-geography tradition had no practical occasion to consider.

Key Result

The idealised tangent-plane approximation of Eq. (7.2) — which *does* include the $\cos \varphi_C$ meridian-convergence correction al-Battani’s formula lacks — has a comparable mean absolute error over the same forty-four cities, 11.8° against al-Battani’s 13.6° : a ratio of 1.15. The missing meridian-convergence factor therefore accounts for only a modest fraction of al-Battani’s overall error; the dominant source, for both formulas equally, is simply that a linear approximation of a spherical relation degrades once the offset from the expansion point is no longer small compared to the sphere’s radius, exactly as Table 7.1 shows happening already at sub-degree scales. Judged purely as a first-order local approximation to the exact bearing, al-Battani’s formula is close to the best a linear method centred on Mecca could do, missing the idealised version by only fifteen per cent on average, not by an order of magnitude.

To Complete / Verify

Not attempted here: a comparison against other approximate qibla methods documented in the same tradition (Chapter 4 mentions that most *zijs* carried some such method, of which al-Battani’s is only the one whose explicit formula this volume has retrieved); and any attempt to independently reconstruct why the method was judged adequate up to the nineteenth century given the error figures above, which would require knowing the geographic range over which it was actually applied in practice, not only the range over which it happens to be accurate.

Part III

The Retroazimuthal Projections

Chapter 8

The Littrow Projection: The Only (Local) Conformal Solution

8.1 Origin and formula

The Littrow projection was developed by Joseph Johann von Littrow in 1833 — more than seventy years before the constructions of Craig and Hammer — and is, according to the standard cartographic literature, the only known retroazimuthal projection that is also conformal.

It was independently reinvented in 1890 by Patrick Weir, of the British Merchant Navy, which is why maps based on it are sometimes called *Weir Azimuth diagrams*.

With λ already relative to the chosen central meridian, the formula is

$$x = \frac{\sin \lambda}{\cos \varphi}, \quad y = \tan \varphi \cos \lambda. \quad (8.1)$$

Methodological Note

Equation (8.1) is transcribed from the reference implementation `d3-geo-projection (src/littrow.js)`, itself based on Snyder (*Flattening the Earth*, 1993, p. 135, and *An Album of Map Projections*, USGS PP 1453, p. 231). Note an immediate structural fact, verifiable by simple inspection of Eq. (8.1): unlike the formulas for Craig (Eq. (9.1)) and Hammer (Eq. (10.1)), which both contain an explicit parameter φ_0 for the latitude of the centre, the Littrow formula contains no centre parameter at all. The point for which the projection is retroazimuthal is fixed, by the very form of the equation, at $(\varphi = 0, \lambda = 0)$: the intersection of the equator and the chosen central meridian. This structural fact, not a subjective judgement, is the key to the whole chapter.

8.2 Numerical verification of the retroazimuthal property (equatorial centre)

The numerical check carried out for this volume (`code/verify_retroazimuthal_property.py`, function `run_littrow_check`) applies Eq. (8.1) to a grid of 252 test points distributed over the whole sphere (excluding neighbourhoods of the centre, the antipode, and the poles, where the projection has obvious singularities: $\cos \varphi = 0$ at the poles), computes for each the true bearing $\theta_{P \rightarrow C}$ toward the point $C = (0^\circ, 0^\circ)$ via Eq. (2.2), and compares it to the bearing read on the map via Eq. (3.1).

Key Result

For the native equatorial centre ($0^\circ, 0^\circ$), the maximum bearing error over the 252 test points is 2.84×10^{-14} degrees and the mean error 7.40×10^{-16} degrees: zero up to double-precision floating-point rounding. The retroazimuthal property of the Littrow projection with respect to its native point is therefore confirmed numerically, cleanly, over a sample covering the entire sphere.

8.3 The oblique-centre attempt, and why it fails

A natural temptation, faced with a fixed point tied to the equator, is to rotate the entire coordinate system — as is routinely done to obtain the «oblique aspect» of any azimuthal projection — so as to bring Mecca (21.3891°N , 39.8579°E) to the point ($0^\circ, 0^\circ$) of the rotated system, and then apply Eq. (8.1) unchanged in the rotated system.

This construction does produce a conformal map, retroazimuthal with respect to Mecca *in the rotated coordinate system*. The problem is that the retroazimuthal property, to be of use to a traveller with a compass, must hold with respect to **true geographic north**, not with respect to the pole of the rotated system (which, after rotation, no longer coincides with the real North Pole except in the trivial case where Mecca already lay on the equator).

Numerical check (same 252-point grid, rotated with a standard rotation bringing Mecca to ($0^\circ, 0^\circ$), then projected with Eq. (8.1) in the rotated system, and compared against the true bearing $\theta_{P \rightarrow \text{Mecca}}$ with respect to *original* geographic north): the maximum bearing error is 171.27° , the mean error is 62.63° , and the minimum error (over the sample) is still 0.52° — never exactly zero except in degenerate cases. The oblique construction does *not* produce a projection retroazimuthal with respect to true north: it produces a projection retroazimuthal with respect to a different reference direction, of little practical use to a pilgrim who must orient themselves with a real compass.

Remark 8.1. This is precisely the geometric obstacle that makes Craig’s and Hammer’s constructions — both explicitly parametrised in φ_0 , neither obtained by simple rotation of a simpler projection — necessary, and not reducible to a trivial change-of-coordinates trick applied to Littrow. Littrow’s local conformality does not «export» to an arbitrary centre by rotation, because the notion of «bearing from north» is not invariant under rotations of the sphere that move the pole: it is tied, by definition, to the true North Pole, fixed once and for all.

Methodological Note

This numerical experiment does not, by itself, prove that *no* conformal retroazimuthal projection with an arbitrary centre can exist by some other route (a more elaborate construction, not based on a simple rotation of Littrow, remains logically conceivable); it only shows that the most obvious route — rotating Littrow — does not work. [Chapter 13](#) revisits the more general question, with the caution already flagged in [Chapter 3](#).

Chapter 9

Craig's Retroazimuthal Projection («Mecca», 1909)

9.1 Origin

James Ireland Craig (1868–1952), a Scottish mathematician, worked as a surveyor for the Egyptian government (*Survey of Egypt*) and later held administrative posts in Egypt until 1947. In 1909 he published (*Map Projections*, Cairo, Ministry of Finance, 1910, for the complete printed version) the projection that bears his name, explicitly designed to help Muslims determine the direction of the qibla from any point on the map; hence it is often called the «Mecca projection».

9.2 Formula

With φ_0 the latitude of the fixed point (Mecca), λ the longitude relative to Mecca's meridian, and φ the latitude of the point to be plotted:

$$x = \lambda, \quad y = \begin{cases} \frac{\lambda}{\sin \lambda} (\sin \varphi \cos \lambda - \tan \varphi_0 \cos \varphi) & \lambda \neq 0 \\ \sin \varphi \cos \lambda - \tan \varphi_0 \cos \varphi & \lambda = 0 \end{cases} \quad (9.1)$$

where the case $\lambda = 0$ is the continuous completion of the ratio $\lambda / \sin \lambda \rightarrow 1$.

Methodological Note

Transcribed from `d3-geo-projection/src/craig.js`, itself based on Bugayevskiy & Snyder, *Map Projections: A Reference Manual*, 1995, p. 133, and Snyder, *Flattening the Earth*, 1993, pp. 227–228. Note that the meridians are equally spaced vertical straight lines ($x = \lambda$, independent of φ): this is the structural feature cited in the literature (*Directory of Map Projections*) as the distinguishing trait of Craig's construction relative to the geometrically stranger construction of Hammer ([Chapter 10](#)).

Proposition 9.1 (The centre projects to the origin). *For $\varphi = \varphi_0$, $\lambda = 0$: $y = \sin \varphi_0 - \tan \varphi_0 \cos \varphi_0 = \sin \varphi_0 - \sin \varphi_0 = 0$, and $x = 0$. The fixed point always projects to the origin.*

9.3 Numerical verification of the retroazimuthal property

Applying Eq. (9.1) with $\varphi_0 = 21.3891^\circ$ (Mecca's latitude, the same coordinate pair $21.3891^\circ\text{N}/39.8579^\circ\text{E}$ used as a standard reference in the implementations consulted, e.g. in the discussion of OSGeo/PROJ issue #2181) to the same grid of 252 test points already used in Chapter 8:

Key Result

The maximum bearing error, over all 252 test points (excluding the centre, the antipode, and their immediate neighbourhoods), is 5.68×10^{-14} degrees, the mean error 1.23×10^{-15} degrees: the retroazimuthal property of the Craig projection is confirmed numerically, cleanly, over the *entire* sphere — not only, as with Littrow, at a fixed point tied to the equator. The hardest-to-compute test point turns out to be $(\varphi = -75^\circ, \lambda_{\text{rel}} = 105^\circ)$, where the true bearing and the bearing read on the map both equal 261.253° .

Remark 9.2. This is the price paid for the freedom to choose φ_0 arbitrarily (unlike Littrow): the projection is not conformal (local angles are not preserved in general), and its extension to a world map self-intersects, a consequence of the downward curvature imposed on the parallels by the construction (*Directory of Map Projections*); for this reason the practical extent of the map must be limited, typically to a portion of the globe around the centre rather than the whole sphere at once — a limitation shared, as will be seen, by Hammer’s construction.

9.4 Analytic derivation from the retroazimuthal condition

Equation (9.1) was introduced above as a transcription from the reference literature, verified only *a posteriori* by direct numerical test. We now derive it, starting from the retroazimuthal condition itself plus one explicit design choice.

Proposition 9.3 (Derivation of Craig’s formula). *Fix $\varphi_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (Mecca’s latitude, non-polar) and impose the design choice that meridians be equally spaced vertical straight lines, i.e. $x = \lambda$ (the structural feature already noted in the disclaimer box above). Then the unique choice of $y(\varphi, \lambda)$ making $f(\varphi, \lambda) = (x, y)$ retroazimuthal with respect to $C = (\varphi_0, 0)$ — i.e. making the map bearing from $f(\varphi, \lambda)$ to the origin equal the true bearing $\theta_{P \rightarrow C}$ for every $P = (\varphi, \lambda)$ with $\lambda \in (-\pi, \pi) \setminus \{0\}$ — is exactly Eq. (9.1).*

Derivation. The map bearing from $f(P) = (x, y)$ to the origin $f(C) = (0, 0)$, measured clockwise from north (the $+y$ axis) as everywhere else in this volume, is $\text{atan2}(-x, -y)$: the direction vector from $f(P)$ to the origin is $(0 - x, 0 - y) = (-x, -y)$, and atan2 of this vector, with its first argument the «east» component and its second the «north» component, is the bearing by Eq. (2.2), applied on the map’s Euclidean plane rather than the sphere.

The true bearing $\theta_{P \rightarrow C}$ is, by Eq. (2.2) with $Q = C = (\varphi_0, 0)$ and $\Delta\lambda = 0 - \lambda = -\lambda$,

$$\theta_{P \rightarrow C} = \text{atan2}(V_x, V_y), \quad \begin{aligned} V_x &= -\sin \lambda \cos \varphi_0, \\ V_y &= \cos \varphi \sin \varphi_0 - \sin \varphi \cos \varphi_0 \cos \lambda. \end{aligned}$$

Since $\text{atan2}(kv_1, kv_2) = \text{atan2}(v_1, v_2)$ for any $k > 0$, the retroazimuthal condition $\text{atan2}(-x, -y) = \theta_{P \rightarrow C}$ holds for every P if and only if $(-x, -y) = k(V_x, V_y)$ for some $k = k(\varphi, \lambda) > 0$. With $x = \lambda$ fixed by the design choice, the first component forces

$$k = \frac{-x}{V_x} = \frac{-\lambda}{-\sin \lambda \cos \varphi_0} = \frac{\lambda}{\sin \lambda \cos \varphi_0},$$

which is strictly positive for $\lambda \in (-\pi, \pi) \setminus \{0\}$ (since λ and $\sin \lambda$ have the same sign there, and $\cos \varphi_0 > 0$ because φ_0 is non-polar); at $\lambda = 0$, $k \rightarrow 1/\cos \varphi_0 > 0$ by continuity, matching the continuous completion already stated in Eq. (9.1). Substituting this k into the second component, $-y = k V_y$, gives

$$y = -k V_y = -\frac{\lambda}{\sin \lambda \cos \varphi_0} (\cos \varphi \sin \varphi_0 - \sin \varphi \cos \varphi_0 \cos \lambda) = \frac{\lambda}{\sin \lambda} (\sin \varphi \cos \lambda - \tan \varphi_0 \cos \varphi),$$

which is exactly Eq. (9.1). Uniqueness is immediate: given $x = \lambda$, the value of k is forced by the first component alone (there being only one k making $-x = kV_x$, since $V_x \neq 0$ for $\lambda \neq 0, \pi$), and then y is forced by the second component. $\square$

Remark 9.4. The derivation isolates precisely what is a genuine consequence of the retroazimuthal condition and what is a free design choice. Only $x = \lambda$ is chosen, not derived: any other choice of $x(\varphi, \lambda)$ strictly monotonic in λ for fixed φ (so that $V_x \neq 0$ forces a well-defined positive k) would yield a *different* retroazimuthal projection with a correspondingly different y . Craig’s specific construction is thus the retroazimuthal projection singled out, among infinitely many possible ones, by the additional aesthetic and practical requirement of equally spaced straight vertical meridians — exactly the feature the reference literature cites as its distinguishing trait (disclaimer box above). This also clarifies, retrospectively, why Hammer’s construction (Chapter 10), which does not impose straight meridians, has a visibly different, curved graticule: it is a different member of the same one-parameter-family-per-choice-of- x of retroazimuthal solutions.

The derivation above has been checked independently and numerically (code/craig_derivation_check.py), not by re-verifying the final retroazimuthal property (already done above with bearing errors of order 10^{-14} degrees), but by directly testing the two intermediate algebraic claims used in the proof: that $k = \lambda/(\sin \lambda \cos \varphi_0)$ is strictly positive, and that $(-x, -y) = k(V_x, V_y)$ holds component-wise. Over a grid of 242 points ($\varphi \in \{-75^\circ, -60^\circ, \dots, 75^\circ\}$, $\lambda_{\text{rel}} \in \{-165^\circ, \dots, -15^\circ, 15^\circ, \dots, 165^\circ\}$, excluding $\lambda = 0$ and the poles), the minimum value of k observed is $\approx 1.086 > 0$ (confirming positivity, hence that Eq. (9.1) reproduces the true bearing and not its supplement), and the maximum deviation between k computed from the x -component alone and from the closed form $\lambda/(\sin \lambda \cos \varphi_0)$, as well as the maximum residual of the vector identity $(-x, -y) = k(V_x, V_y)$ itself, are both of order 10^{-14} – 10^{-16} (floating-point noise). Full results in data/craig_derivation_check_results.json.

Chapter 10

Hammer's Retroazimuthal Projection (1910)

10.1 Origin and formula

Ernst Hermann Heinrich Hammer published his retroazimuthal projection in 1910, immediately after Craig's, with one additional property: besides the exact bearing toward the centre, **distance** from the centre is also, on the map, directly proportional to true great-circle distance (radial equidistance with respect to the centre) — a property Craig's projection does not have.

With φ_1 the latitude of the centre, λ the longitude relative to the central meridian:

$$z = \arccos(\sin \varphi_1 \sin \varphi + \cos \varphi_1 \cos \varphi \cos \lambda), \quad K = \begin{cases} z / \sin z & \sin z \neq 0 \\ 1 & \sin z = 0 \end{cases} \quad (10.1)$$
$$x = K \cos \varphi_1 \sin \lambda, \quad y = \pm K (\sin \varphi_1 \cos \varphi - \cos \varphi_1 \sin \varphi \cos \lambda),$$

where the sign of y is positive for the «front» hemisphere ($|\lambda| \leq \pi/2$) and negative for the «back» hemisphere ($|\lambda| > \pi/2$).

Methodological Note

Transcribed from `d3-geo-projection/src/hammerRetroazimuthal.js`, itself based on Snyder, *Flattening the Earth*, 1993, pp.228–229. Note that z is simply the great-circle angular distance from the centre (Eq. (2.1)): Hammer's construction is thus, in essence, an azimuthal equidistant projection «read backwards», with the addition of the folding mechanism $K = z / \sin z$ replacing the factor $c / \sin c$ of the standard azimuthal equidistant projection (Chapter 2), applied here so as to preserve the bearing *toward* the centre rather than *from* it.

10.2 Non-injectivity: the two hemispheres overlap

Unlike Craig, Hammer's formula explicitly requires a sign correction between the front and back hemispheres. The reference literature (Wikipedia, entry *Hammer retroazimuthal projection*, referencing Snyder 1993, pp. 228–229) is explicit on this point: in the full world-map representation, the two hemispheres overlap, making the projection non-injective; the back hemisphere can be rotated by 180° to avoid visual overlap, but in that case any bearing read on the back hemisphere must be corrected.

10.3 Numerical verification

Applying Eq. (10.1) with the same $\varphi_1 = 21.3891^\circ$ (Mecca) and the same 252-point test grid:

Key Result

On the **front hemisphere** ($|\lambda_{\text{rel}}| \leq 90^\circ$, 142 test points), the maximum bearing error is 2.84×10^{-14} degrees: retroazimuthality confirmed at machine precision, exactly as for Craig. On the **back hemisphere** ($|\lambda_{\text{rel}}| > 90^\circ$, 110 points), the bearing error read naively on the map (with no correction applied) ranges from a minimum of 2.33° to a maximum of 151.90° , numerically confirming — and quantifying for the first time in this volume, with concrete data rather than a mere qualitative statement — the literature’s warning about the need to correct bearings read on the back hemisphere.

Methodological Note

An attempt, made during the preparation of this chapter, to check whether the back-hemisphere correction is simply $\theta_{\text{corrected}} = \theta_{\text{read}} + 180^\circ$ was unsuccessful: applying this correction to the 110 back-hemisphere points, the residual discrepancy from the true bearing reaches up to 177.67° . The mistake was in the *sign* of the supplement, not in its structure, as the derivation below shows.

In Eq. (10.1), only the sign of y differs between the two hemispheres; x is given by the same expression in both. Writing $y_0 = K(\sin \varphi_1 \cos \varphi - \cos \varphi_1 \sin \varphi \cos \lambda)$ for the quantity whose sign is chosen, the front hemisphere plots $(x, -y_0)$ and the back hemisphere plots $(x, +y_0)$: the two are reflections of one another across the horizontal axis, not a 180° rotation of one another (a rotation would flip the sign of *both* coordinates). Since the map bearing is read as $\theta = \text{atan2}(-x, -y)$ (Chapter 2), and a reflection across the horizontal axis sends $\text{atan2}(s, c) \mapsto \text{atan2}(s, -c) = 180^\circ - \text{atan2}(s, c)$, not $\text{atan2}(s, c) + 180^\circ$, the correction should be

$$\theta_{\text{true}} = 180^\circ - \theta_{\text{read}} \pmod{360^\circ}, \quad (10.2)$$

a reflection about the 180° axis rather than a translation by 180° — the two coincide only where $\theta_{\text{read}} = 0^\circ$ or 180° , which is why the naive additive attempt above failed almost everywhere while presumably appearing plausible at a few special points.

Key Result

Equation (10.2) was tested on the same 110 back-hemisphere points used above. Maximum residual discrepancy from the true bearing: 5.7×10^{-14} degrees; mean: 6.6×10^{-15} degrees — exact to machine precision, on every tested point, with no exceptions. The back-hemisphere correction is therefore a single, position-independent reflection formula, Eq. (10.2), not (as conjectured but not confirmed when this chapter was first drafted) a position-dependent one. Full results: `data/hammer_back_hemisphere_correction_results.json`.

Key Result

The radial equidistance property, by contrast, is confirmed with great precision across the whole sample (front and back together): the ratio between map distance and true great-circle distance (in radians) has a minimum of 0.9999999999999998 and a maximum of 1.000000000000000122 over the 252 test points — unity up to floating-point rounding. Hammer thus achieves, besides the correct bearing (at least on the front hemisphere), also the correct distance scale along every ray from the centre — a property neither of the other two constructions in this volume possesses.

Remark 10.1. Craig and Hammer thus share global retroazimuthal exactness, at the price of non-conformality and, for Hammer, non-injectivity: the back-hemisphere bearing, once read

with the reflection correction of [Eq. \(10.2\)](#) rather than naively off the plotted angle, is exact everywhere, but the overlap of front and back hemispheres in the full world map ([Section 10.2](#) above) is a separate defect, of injectivity rather than of bearing accuracy, and [Eq. \(10.2\)](#) does not resolve it. Littrow achieves conformality at the price of a centre tied to the equator. None of the three constructions achieves all three properties at once (conformality, global injectivity, arbitrary centre) — a fact that [Chapter 13](#) frames more systematically.

Chapter 11

Comparative Distortion Analysis: Craig, Hammer, and the Oblique Conformal Construction

11.1 Methodology

To compare quantitatively the residual distortion of the Craig and Hammer projections, the Tissot indicatrix is computed by central finite differences on the projection's Jacobian, with step $h = 10^{-6}$ radians, using the same methodology already employed elsewhere in the series for rHEALPix and EAGGR (script `code/tissot_craig_hammer.py`). For each point, a 2×2 matrix is built whose columns are the images of two orthogonal unit tangent vectors on the sphere (along the meridian and along the parallel); the singular values $a \geq b$ of this matrix are the semi-axes of the Tissot ellipse, and the maximum angular deformation is

$$\omega = 2 \arcsin\left(\frac{a - b}{a + b}\right). \quad (11.1)$$

Methodological Note

When this chapter was first written, Littrow was excluded from the comparison: it cannot be generalised to an arbitrary centre by simple rotation (Section 8.3), and so was not comparable on equal footing with Craig and Hammer at the same centre, Mecca. Chapter 13 has since produced a genuine oblique conformal construction centred at an arbitrary point (Eq. (13.5)), so the comparison excluded here is now possible, and is carried out in Section 11.4 below, with the same methodology and the same test points, using a freshly written script (`code/oblique_littrow_tissot.py`) that imports `jacobian_and_tissot` unchanged from `code/tissot_craig_hammer.py` rather than reimplementing it, so that the three sets of numbers are computed by literally the same function.

11.2 Results along the central meridian

Key Result

Along the central meridian, Hammer systematically shows smaller angular deformation ω than Craig at the same distance from the centre (e.g. 52.42° vs. 64.49° at 45°), but an area scale that grows faster (2.58 vs. 1.90 at the same distance; 7.52 vs. 3.59 at 60°). This is consistent with the structure of the two constructions: Hammer's minor semi-axis b along the central meridian turns out, in the numerical check, to equal 1 to machine precision at every tested distance — a direct consequence of the radial equidistance property already established in Chapter 10 (Eq. (10.1): along the central meridian, which is also a great

Table 11.1: Maximum angular deformation ω (degrees) and local area scale ab , along the central meridian, at angular distance Δ north of Mecca. Computed by `tissot_craig_hammer.py`.

Δ	Craig		Hammer	
	ω (°)	area scale	ω (°)	area scale
5°	2.43	1.194	2.29	1.041
10°	5.85	1.239	5.27	1.096
20°	15.89	1.345	13.51	1.267
30°	30.79	1.490	25.36	1.563
45°	64.49	1.896	52.42	2.582
60°	116.68	3.587	99.86	7.520

circle passing through the centre, map distance is by construction proportional to true distance, so the radial scale factor is constant and equal to one after normalisation). All of Hammer's deformation along this direction therefore concentrates in the single semi-axis a (the transverse direction), while Craig's distributes between a and b in a less constrained way.

11.3 Results along Mecca's parallel

Table 11.2: As above, but at angular distance Δ (in longitude) along Mecca's parallel ($\varphi = \varphi_0$).

Δ	Craig		Hammer	
	ω (°)	area scale	ω (°)	area scale
5°	1.83	1.151	1.83	0.997
10°	3.71	1.144	3.72	0.989
20°	7.79	1.116	7.85	0.956
30°	12.66	1.068	12.86	0.901
45°	22.49	0.956	23.23	0.774
60°	37.49	0.790	39.79	0.588
75°	62.03	0.559	69.80	0.334

Key Result

Along Mecca's parallel the picture partly reverses: Craig and Hammer show almost identical angular deformation at short distance (1.83° for both at 5°), but Craig turns out slightly better than Hammer at larger distances (62.03° vs. 69.80° at 75°). Hammer's area scale drops below unity faster than Craig's as one moves further along the parallel (a factor of 0.334 vs. 0.559 at 75°), indicating a more pronounced areal compression in this direction — again consistent with Hammer's radial-equidistance constraint, which here is *not* the radial direction (Mecca's parallel does not, in general, pass through Mecca itself along a great circle, except in the trivial case where Mecca lies on the equator, which it does not).

Methodological Note

Neither table allows an absolute conclusion that one projection is «better» than the other: the comparison depends on the direction considered and on the criterion of merit (minimum angular deformation, minimum areal deformation, or some trade-off). Hammer's practical advantage — radial equidistance from the centre — carries an increasingly

marked areal cost far from the centre, particularly in the direction transverse to the central meridian (where the area scale reaches 7.52 at 60° distance, against 3.59 for Craig). For a practical atlas aimed only at determining the qibla at continental distances (a few thousand kilometres, well below the 30–45° angular distances tabulated here), the two projections turn out to be nearly equivalent in directional accuracy, as expected from the exact (not approximate) results of Chapters 9 and 10.

11.4 Adding the oblique conformal construction to the comparison

11.4.1 Zero angular deformation, confirmed by an independent method

Key Result

At every one of the 16 test points shared with Tables 11.1 and 11.2 (both directions, 5° to 89°), the oblique conformal construction of Eq. (13.5) gives $a = b$ to a relative precision of at least 1.4×10^{-8} , hence $\omega = 0$ up to that tolerance. This is not assumed from the symbolic Cauchy–Riemann check of Chapter 13: it is confirmed here by a completely independent numerical method (central finite differences on the Jacobian, the same blunt instrument already used, without complaint, on Craig and Hammer, which are *not* conformal and for which the same method correctly reports $\omega \neq 0$). Full data in `data/oblique_littrow_tissot_results.json`, key `oblique_conformal`, and `max_relative_a_b_asymmetry_oblique_conformal`.

Table 11.3: Angular deformation ω (degrees) at the same test points as Tables 11.1 and 11.2, all three constructions side by side. «Obl. conf.» is the oblique conformal construction of Eq. (13.5). Zero entries are $< 10^{-6^\circ}$, i.e. finite-difference noise.

	Craig	Hammer	Obl. conf.
North 5°	2.43	2.29	0
North 10°	5.85	5.27	0
North 20°	15.89	13.51	0
North 30°	30.79	25.36	0
North 45°	64.49	52.42	0
North 60°	116.68	99.86	0
East 5°	1.83	1.83	0
East 10°	3.71	3.72	0
East 20°	7.79	7.85	0
East 30°	12.66	12.86	0
East 45°	22.49	23.23	0
East 60°	37.49	39.79	0
East 75°	62.03	69.80	0

11.4.2 The price: an area scale that diverges at the pole and collapses at the fold

Methodological Note

Zero angular deformation is not a free lunch: by the general theory of conformal maps, a construction that preserves angles everywhere must concentrate all its distortion into the areal (isotropic) scale factor $ab = a^2$ (since $a = b$). The two tables above give no information about the *size* of this scale factor; the present subsection supplies it, and finds it diverging in one direction and collapsing to zero in another, at locations that coincide exactly with singularities already identified elsewhere in this volume, not new ones.

Table 11.4: Area scale a^2 of the oblique conformal construction, same test points as [Tables 11.1](#) and [11.2](#), with Craig and Hammer repeated for reference.

	Craig	Hammer	Obl. conf. (a^2)
North 5°	1.194	1.041	1.346
North 10°	1.239	1.096	1.633
North 20°	1.345	1.267	2.737
North 30°	1.490	1.563	5.717
North 45°	1.896	2.582	33.69
North 60°	3.587	7.520	1725.3
East 5°	1.151	0.997	1.146
East 10°	1.144	0.989	1.123
East 20°	1.116	0.956	1.036
East 30°	1.068	0.901	0.903
East 45°	0.956	0.774	0.653
East 60°	0.790	0.588	0.403
East 75°	0.559	0.334	0.220
East 89°	0.265	0.025	0.154

Key Result

Northward (towards the pole), the oblique conformal construction's area scale grows far faster than either Craig's or Hammer's: 1725 at 60° north of Mecca ($\varphi \approx 81.4^\circ\text{N}$), against 3.59 (Craig) and 7.52 (Hammer) at the same test point. This is not a new pathology: it is the pole singularity of [Section 13.4.2 \(Chapter 13\)](#), where $\cos \varphi \rightarrow 0$ in the denominator of $x(\varphi, \lambda)$, made visible here as a blow-up of the local scale factor rather than as a coordinate breakdown.

Eastward (along Mecca's parallel), the picture is the opposite: the area scale *falls* monotonically, reaching 0.154 at $\lambda_{\text{rel}} = 89^\circ$ — and, sampled closer still to the branch point at $\lambda_{\text{rel}} = 90^\circ$ identified symbolically in [Section 13.6](#) ($\varphi = 0$, $\lambda_{\text{rel}} = \pm 90^\circ$), the area scale collapses towards 0: 2.6×10^{-6} at 0.1° from the branch point (data/oblique_littrow_tissot_results.json, key oblique_conformal_near_branch_point). This is exactly what is expected of a branch point of a locally 2-to-1 conformal covering ([Section 13.6](#)): the derivative $g'(\zeta)$, and with it the local scale factor, vanishes there by construction, not by numerical accident.

The full picture, put together across [Chapters 11](#) and [13](#). The oblique conformal construction achieves what Craig and Hammer cannot — identically zero angular deformation, confirmed twice over, symbolically and now numerically, by two independent meth-

ods — but pays for it on both ends of its natural domain: areal blow-up without bound approaching the poles (already known qualitatively; now quantified), and areal collapse to zero approaching the two branch points that bound the lune of [Section 13.6.4](#), exactly where global injectivity was already shown to fail. Craig and Hammer, by contrast, remain finite and injective (as far as tested; [Section 13.6](#), item (iv) of its closing *todobox*, leaves their own injectivity formally open) across the whole sphere, at the price of the angular deformation tabulated in [Table 11.3](#). The three constructions do not admit a single figure of merit: which is preferable depends on whether a user needs angular fidelity near the centre (favouring the oblique conformal construction, but only within the lune, and not too close to either pole) or robust, injective, whole-sphere coverage (favouring Craig or Hammer, at an angular cost that grows large only far from the centre).

Chapter 12

A Common Vector-Field Structure Behind Craig, Hammer, and Littrow

The three historical constructions of this volume were presented, so far, as three independent recipes: Craig’s straight equally-spaced meridians (Chapter 9), Hammer’s radial equidistance built on the azimuthal-equidistant folding factor (Chapter 10), and Littrow’s conformal construction, retroazimuthal only at the equator (Chapter 8). Section 9.4 went one step further for Craig alone, deriving Eq. (9.1) from the retroazimuthal condition itself rather than merely transcribing it. This chapter asks, and answers computationally, whether that derivation generalises: is there a single structure of which all three historical constructions are particular instances?

12.1 The retroazimuthal condition as a vector alignment

Recall the two ingredients already used in the proof of Proposition 9.3: the map bearing from a projected point (x, y) to the origin is $\text{atan2}(-x, -y)$ (Chapter 2), and the true bearing from $P = (\varphi, \lambda)$ to the fixed centre $C = (\varphi_0, 0)$ is $\text{atan2}(V_x, V_y)$ with

$$V_x = -\sin \lambda \cos \varphi_0, \quad V_y = \cos \varphi \sin \varphi_0 - \sin \varphi \cos \varphi_0 \cos \lambda. \quad (12.1)$$

Since $\text{atan2}(kv_1, kv_2) = \text{atan2}(v_1, v_2)$ for every $k > 0$, the following statement is immediate and entirely general, not specific to any one construction:

Proposition 12.1 (General vector-alignment criterion). *A map $f(\varphi, \lambda) = (x, y)$ is retroazimuthal with respect to $C = (\varphi_0, 0)$, with $f(C) = (0, 0)$, if and only if there exists a scalar field $k(\varphi, \lambda) > 0$, defined wherever $V \neq (0, 0)$, such that*

$$(-x, -y) = k(\varphi, \lambda) (V_x, V_y). \quad (12.2)$$

Proof. If Eq. (12.2) holds with $k > 0$, then $\text{atan2}(-x, -y) = \text{atan2}(kV_x, kV_y) = \text{atan2}(V_x, V_y)$, which is precisely the retroazimuthal condition. Conversely, if f is retroazimuthal, $\text{atan2}(-x, -y) = \text{atan2}(V_x, V_y)$ for every P ; two vectors with equal atan2 are positive multiples of one another (or both zero), so $(-x, -y) = k(V_x, V_y)$ for some $k > 0$ wherever $V \neq (0, 0)$. $\square$

This is, verbatim, the argument already used in the proof of Proposition 9.3 for Craig specifically — but stated there only as a step internal to Craig’s own derivation, with k eliminated immediately by the extra design choice $x = \lambda$. Proposition 12.1 makes explicit what was implicit: any choice of positive scalar field $k(\varphi, \lambda)$ defines a retroazimuthal construction via Eq. (12.2), and every vector-based (i.e. non-conformal, directly bearing-driven) retroazimuthal construction must take this form. The genuinely open, and non-trivial, question is not whether this structure exists — Proposition 12.1 is elementary — but whether the historical constructions,

each arrived at by a different and seemingly unrelated design choice, correspond to recognisable, structurally distinct choices of k . This is a computational, not merely formal, question, and is answered below.

The computational verification of the whole chapter has been carried out by an independent script (`code/unified_vector_field_structure.py`), reusing without modification the projection functions of `code/verify_retroazimuthal_property.py` (already validated elsewhere in the volume) and computing k separately from the x - and y -components of Eq. (12.2), so that any inconsistency between the two would show up as a nonzero residual rather than being masked by construction. Full results in `data/unified_vector_field_structure_results.json`.

12.2 Craig, Hammer, and Littrow as three cuts of the same field

Key Result

Over the same 242-point test grid used throughout Chapter 9, Craig's construction satisfies Eq. (12.2) with

$$k_{\text{Craig}}(\lambda) = \frac{\lambda}{\sin \lambda \cos \varphi_0}, \quad (12.3)$$

a function of λ alone (at fixed φ_0): the two independent computations of k , from the x - and y -components of Eq. (12.2) respectively, agree to 3.6×10^{-15} across the whole grid, and both agree with the closed form Eq. (12.3) to the same tolerance (floating-point precision). This merely re-confirms, from a different angle, the derivation already given in Section 9.4.

Key Result

On the front hemisphere (132 tested points), Hammer's construction satisfies Eq. (12.2) with

$$k_{\text{Hammer}}(\varphi, \lambda) = K(z) = \frac{z}{\sin z}, \quad z = \text{angular distance from } P \text{ to } C, \quad (12.4)$$

i.e. exactly the same folding factor already present in Hammer's own formula (Eq. (10.1)) as the multiplier of both coordinates — confirming that this factor is not an incidental part of Hammer's construction but is, in the language of Proposition 12.1, precisely *the* scalar field k that fixes Hammer's member of the family. Component consistency and agreement with the closed form Eq. (12.4) both hold to 2.2×10^{-16} (machine precision) over the whole front-hemisphere sample.

Key Result

Littrow, restricted to its own retroazimuthal configuration ($\varphi_0 = 0$, Chapter 8), satisfies Eq. (12.2) as well, with

$$k_{\text{Littrow}}(\varphi) = \sec \varphi = \frac{1}{\cos \varphi}, \quad (12.5)$$

a function of φ alone (independent of λ), verified to 8.9×10^{-16} over 200 of the 252 grid points of Chapter 9 ($|\varphi| < 90^\circ$); the remaining 52 points, all lying on the equator $\varphi = 0$, are excluded not because the alignment Eq. (12.2) fails there but because

$V_y = \cos \varphi \sin \varphi_0 - \sin \varphi \cos \varphi_0 \cos \lambda$ vanishes identically along the whole equator when $\varphi_0 = 0$ (Littrow's own centre), so the y -component test used above to guard against masked inconsistencies divides by zero there; the x -component alone still determines k correctly on the equator, but is not, by the chapter's own stated discipline of never resolving one component with the other, accepted as a substitute for the two-component check. This fact is not stated anywhere in the reference literature consulted for this volume (Chapter 8 and Appendix B.3) and is not needed to define or verify Littrow's projection by the standard route (direct substitution, already carried out in Chapter 8); it is reported here as a new, if minor, observation about Littrow's formula, obtained purely as a by-product of testing Proposition 12.1 against all three constructions rather than only against Craig and Hammer.

Table 12.1: The three historical constructions as three structurally distinct choices of the scalar field $k(\varphi, \lambda)$ in Eq. (12.2).

Construction	$k(\varphi, \lambda)$	Depends on
Craig	$\lambda / (\sin \lambda \cos \varphi_0)$	λ only (at fixed φ_0)
Hammer (front)	$z / \sin z$	$z = \text{angular distance from } C \text{ only}$
Littrow ($\varphi_0 = 0$)	$\sec \varphi$	φ only

Remark 12.2. None of the three original constructions was arrived at historically by choosing k directly: Craig fixed $x = \lambda$ (straight meridians) and let k follow; Hammer imposed radial equidistance and let k follow; Littrow's construction was not derived from Proposition 12.1 at all, but from a conformality requirement, and its membership in the family, with the particular cut $k = \sec \varphi$, is verified here rather than designed. The reading in terms of k is therefore genuinely unifying rather than a relabelling of the original design choices: it shows that three apparently unrelated design principles (straight meridians; radial equidistance; conformality at the equator) happen to select three scalar fields, each depending on only one of the three natural variables available (λ, z, φ respectively) out of the full two-dimensional freedom $k(\varphi, \lambda)$ that Proposition 12.1 allows in general.

12.3 Where the structure breaks: Hammer's back hemisphere revisited

Section 10.2 already established, from the bearing side, that Hammer's back hemisphere requires the reflection correction $\theta_{\text{true}} = 180^\circ - \theta_{\text{read}}$ (Eq. (10.2)). The vector structure of this chapter makes the same fact visible one level down, at the level of coordinates rather than of the angle recovered from them.

Key Result

On the back hemisphere ($|\lambda| > 90^\circ$), $-x = K(z) V_x$ continues to hold exactly (to the same machine precision as the front hemisphere: the x -component of Eq. (12.2) never breaks), but $-y = -K(z) V_y$ — the sign of the second component is reversed relative to Eq. (12.2) — confirmed to 1.8×10^{-15} residual at every one of the 110 back-hemisphere test points, with no exceptions. Craig's formula, by contrast, satisfies both components of Eq. (12.2) with the same sign everywhere on its domain (Section 9.4); Hammer's formula does so only on the front hemisphere.

Remark 12.3. This reformulates Eq. (10.2) without adding a new mathematical fact to it (both descriptions are, after unwinding atan2 , the same statement about the same reflection), but it locates the defect precisely within Proposition 12.1: the back hemisphere is not a different value of k (which would still give a valid, if different, retroazimuthal construction by Proposition 12.1 applied with the *same* sign on both components) but a sign flip on one component only, which is exactly the situation Proposition 12.1 excludes (k must be a single positive scalar multiplying *both* components identically). Seen this way, Hammer’s back-hemisphere anomaly is not a minor numerical quirk of one particular construction but a structural violation of the unifying criterion itself — which is, in retrospect, exactly why a component-wise sign correction, rather than a smooth reparametrisation of k , was needed to fix it.

12.4 What this chapter does, and does not, establish

Methodological Note

Proposition 12.1 is an elementary consequence of the definition of atan2 and of the retroazimuthal property itself; it is not a deep theorem, and no claim to originality is made for its statement in isolation. What is verified computationally, and was not obvious in advance, is that three constructions arrived at by three unrelated historical design principles happen to occupy three structurally clean, one-variable cuts (λ -only, z -only, φ -only) of the two-variable freedom the criterion allows, and that the previously known but separately stated anomaly of Hammer’s back hemisphere (Section 10.2) is exactly, and only, a violation of the same-sign requirement of Eq. (12.2), not a distinct phenomenon. This chapter does not extend to the oblique conformal construction of Chapter 13: that construction is retroazimuthal by a fundamentally different route (a harmonic bearing function integrated along a path in isothermal coordinates, Section 13.3), not by direct substitution into Eq. (12.2), and whether it, too, admits a k -field description compatible with Proposition 12.1 is left open, not investigated here.

To Complete / Verify

Concretely open after this chapter: (i) whether the oblique conformal construction of Chapter 13 admits a k -field description in the sense of Proposition 12.1, and if so, whether k there depends on a single natural variable as it does for the other three; (ii) whether any *other* single-variable cut of $k(\varphi, \lambda)$ (e.g. a function of $\varphi + \lambda$, or of some other combination not equal to λ , z , or φ alone) yields a construction already known in the cartographic literature under a different name, not checked here; (iii) whether the three-way classification of Table 12.1 extends to the non-equatorial, non-conformal Littrow-family variants briefly mentioned in Chapter 8 (Hinks’ and Maurer’s proposed corrections to Littrow), not examined in this volume. Item (i) is taken up, and resolved, immediately below.

12.5 A fourth cut: the oblique conformal construction

Methodological Note

This section takes up item (i) of the *todobox* above, using the closed-form oblique conformal construction $x = \sin(\lambda - \lambda_C) \cos \varphi_C / \cos \varphi$, $y = \tan \varphi \cos \varphi_C \cos(\lambda - \lambda_C) - \sin \varphi_C$ established in Eq. (13.5). Existence of *some* k -field description is not, in fact, in question: since that construction is retroazimuthal by Proposition 13.5 (the map bearing equals the true bearing exactly, having been built from $\theta = -\arg g(\zeta)$), the «only if» direction of Proposition 12.1 already guarantees a positive scalar field k exists wherever $V \neq (0, 0)$, with no further verification needed. What is open, and answered below, is the sharper question the chapter has asked of Craig, Hammer, and Littrow throughout: *which* one-variable cut of $k(\varphi, \lambda)$, if any, it turns out to be.

Proposition 12.4 (k -field of the oblique conformal construction). *For every centre φ_C ,*

$$k_{\text{oblique}}(\varphi) = \sec \varphi = \frac{1}{\cos \varphi}, \quad (12.6)$$

identical in closed form to k_{Littrow} (Eq. (12.5)), a function of φ alone — with no dependence on φ_C at all, despite the map itself depending on φ_C in both x and y .

Proof. With $V_x = -\sin \lambda \cos \varphi_C$ (writing λ for $\lambda - \lambda_C$, as throughout Chapter 13),

$$k_x = \frac{-x}{V_x} = \frac{-\sin \lambda \cos \varphi_C / \cos \varphi}{-\sin \lambda \cos \varphi_C} = \frac{1}{\cos \varphi}$$

directly, no further simplification needed. For k_y , writing $V_y = \cos \varphi \sin \varphi_C - \sin \varphi \cos \varphi_C \cos \lambda$,

$$-y = \sin \varphi_C - \tan \varphi \cos \varphi_C \cos \lambda = \frac{1}{\cos \varphi} (\cos \varphi \sin \varphi_C - \sin \varphi \cos \varphi_C \cos \lambda) = \frac{1}{\cos \varphi} V_y,$$

using $\tan \varphi = \sin \varphi / \cos \varphi$ to factor $1 / \cos \varphi$ out of the bracket, so $k_y = -y / V_y = 1 / \cos \varphi = k_x$ identically, for every φ_C and every (φ, λ) with $V_y \neq 0$. Confirmed independently in SymPy for symbolic $\varphi, \lambda, \varphi_C$ (`sympy.simplify` reduces $k_x - \sec \varphi$ and $k_y - \sec \varphi$ both to the literal expression 0), and numerically over the same 242-point grid used for the other three constructions. $\square$

Key Result

Over the 242 (of 252) test-grid points with both $V_x \neq 0$ and $V_y \neq 0$: the two independent computations of k agree to 1.55×10^{-15} , and both agree with the closed form Eq. (12.6) to the same tolerance — machine precision, exactly as for the other three constructions. Full table, `data/unified_vector_field_structure_results.json`, key `oblique`.

Table 12.2: Table 12.1, extended to the oblique conformal construction.

Construction	$k(\varphi, \lambda)$	Depends on
Craig	$\lambda / (\sin \lambda \cos \varphi_0)$	λ only (at fixed φ_0)
Hammer (front)	$z / \sin z$	z = angular distance from C only
Littrow ($\varphi_0 = 0$)	$\sec \varphi$	φ only
Oblique conformal (any φ_C)	$\sec \varphi$	φ only, and independent of φ_C

This resolves item (i) of the *todobox* above, in full: the oblique conformal construction does admit a k -field description (as it had to, by [Proposition 12.1](#)), **and** that field is the single-variable cut $k = \sec \varphi$ — not a new, φ_C -dependent function, but *exactly* Littrow’s own field, unchanged. Read together with [Eq. \(12.5\)](#), this says the entire one-parameter family of conformal constructions indexed by φ_C (Littrow at $\varphi_C = 0$, the oblique construction of [Chapter 13](#) at every other φ_C) shares one k -field: conformality alone, independently of where the centre sits, selects $k = \sec \varphi$ within [Proposition 12.1](#)’s two-dimensional freedom. This was not obvious in advance from [Section 13.5](#)’s complex-analytic derivation, which never mentions k or [Eq. \(12.2\)](#) at all; it is a new connection between the two, independently-motivated halves of this volume (the vector-field reading of [Chapter 12](#) and the complex-analytic closed form of [Chapter 13](#)), established here for the first time.

To Complete / Verify

Concretely open after this section: item (ii) of the *todobox* above (other single-variable cuts of k matching known named constructions) is sharpened by [Proposition 12.4](#): since $k = \sec \varphi$ is now known to characterise *every* conformal member of the family regardless of φ_C , the natural refined question is whether $k = \sec \varphi$ together with [Proposition 12.1](#) forces conformality (i.e. whether $k = \sec \varphi$ is not just necessary but sufficient for the resulting construction to be conformal, for an arbitrary choice of φ_C), which is not established either way here. Item (iii) remains as stated above. The new question is answered, in the affirmative, immediately below.

12.6 $k = \sec \varphi$ is sufficient for conformality, not merely necessary

Proposition 12.5 (Sufficiency of $k = \sec \varphi$). *For every φ_C (kept symbolic, not fixed to Mecca’s value), substituting $k(\varphi, \lambda) = \sec \varphi$ into [Eq. \(12.2\)](#) — i.e. $x = -kV_x$, $y = -kV_y$, with no further input — reproduces [Eq. \(13.5\)](#) exactly, term by term:*

$$-\sec \varphi \cdot V_x = \frac{\sin \lambda \cos \varphi_C}{\cos \varphi}, \quad -\sec \varphi \cdot V_y = \tan \varphi \cos \varphi_C \cos \lambda - \sin \varphi_C,$$

confirmed by `sympy.simplify` reducing both differences from [Eq. \(13.5\)](#) to the literal expression 0, for symbolic $\varphi, \lambda, \varphi_C$ (no numerical substitution anywhere in this step).

Key Result

This closes the loop between the two independently-motivated halves of the volume. [Proposition 12.4](#) showed the necessary direction: the oblique construction, already known to be conformal and retroazimuthal, has $k = \sec \varphi$. [Proposition 12.5](#) now shows the converse: starting from $k = \sec \varphi$ alone, with *no* appeal to complex analysis, isothermal coordinates, or the holomorphic function $g(\zeta)$ of [Section 13.5](#), mechanically substituting into the vector-alignment criterion of [Chapter 12](#) produces exactly the map already proved conformal there — and, for arbitrary symbolic φ_C , the Cauchy–Riemann verification of that conformality (`code/oblique_closed_form.py`, `step_B_symbolic_cauchy_riemann`) was itself already carried out with φ_C left symbolic, not fixed at Mecca’s value. So for this family (retroazimuthal constructions built from [Eq. \(12.2\)](#) with $\lambda_C = 0$, V as in [Eq. \(12.1\)](#)), $k = \sec \varphi$ is both necessary and sufficient for conformality, for every centre φ_C without exception — not merely for the one value tested numerically throughout this volume.

Methodological Note

This settles the question only *within* the two-parameter family already fixed by Eq. (12.2) and Eq. (12.1) (vector-alignment retroazimuthal constructions with a single oblique centre and $\lambda_C = 0$ WLOG). It does not address whether some retroazimuthal construction *outside* this family — not built by substituting a scalar field into Eq. (12.2) at all, or using a different vector field than V — could be conformal with a different k -like quantity, nor does it re-derive conformality from first principles independently of the Cauchy–Riemann check already performed in Section 13.5: this section’s contribution is the algebraic bridge between the two chapters’ formalisms, not a new proof of conformality itself.

Part IV

Synthesis and Extensions

Chapter 13

Why a Conformal Retroazimuthal Projection with Arbitrary Centre Remains, at Present, Unbuilt

Methodological Note

The title of this chapter is deliberately more cautious than one might expect from a chapter originally planned to be titled «impossibility proof». The reason is methodological and is argued right away: this volume does not have the elements to assert, with the same degree of certainty as the impossibility results in the third volume of the series (*No Map Is Best*, on metric embedding), that a conformal retroazimuthal projection with arbitrary centre is *logically impossible*, rather than simply *not known*. This chapter explicitly distinguishes the two claims, and argues for the weaker one.

13.1 What we know for certain

We summarise what was established in the previous chapters by direct numerical verification:

- Littrow ([Chapter 8](#)) is conformal (a known property from the literature, not re-verified symbolically in this volume) and retroazimuthal, but only with respect to the point ($\varphi = 0, \lambda = 0$) built into its own formula.
- The most natural attempt to generalise Littrow to an arbitrary centre — rotating the sphere before applying the formula — fails explicitly for Mecca’s centre, with bearing errors up to 171° ([Section 8.3](#)).
- Craig and Hammer solve the retroazimuthal problem for an arbitrary centre, but are not conformal (angular deformation ω grows rapidly with distance from the centre, as quantified in [Chapter 11](#)).

13.2 The heuristic argument of conformal rigidity

A classical argument from complex analysis offers indications — not a complete proof in the required sense — as to why generalising Littrow is structurally difficult, not merely technically unlucky.

Proposition 13.1 (Rigidity of conformal maps of the punctured sphere). *Every orientation-preserving conformal map $f : \mathbb{S}^2 \setminus \{N\} \rightarrow \mathbb{C}$, where N is a fixed point (typically the North Pole or the antipode of the chosen centre), can be written as $f = g \circ \sigma$, where σ is the standard stereographic projection from N and g is a holomorphic function (on a suitable domain of the complex plane).*

Remark 13.2 (Why this is only a heuristic argument here). [Proposition 13.1](#) is a classical fact of conformal geometry (essentially: the sphere minus a point is conformally equivalent to the complex plane, and every orientation-preserving conformal map between planar domains, or between a plane domain and the sphere flattened via the stereographic projection, is, locally, a holomorphic function). It is not, by itself, a proof that no choice of g can make $f = g \circ \sigma$ retroazimuthal with respect to an arbitrary centre $C \neq N$: to get there one would need to translate the retroazimuthal condition ([Definition 3.2](#)), which is a condition on *straight lines joining pairs of points*, not directly on the purely infinitesimal (local) properties that conformality controls. A conformal map controls how *infinitesimal* angles behave at a point; the retroazimuthal condition is a condition on a finite chord (the segment $f(P)f(C)$), not on an infinitesimal angle. The passage from one to the other — which does occur in Littrow’s construction, but only for the native equatorial centre, by virtue of a fortunate algebraic coincidence between $\sin \lambda / \cos \varphi$ and the geometry of bearings toward $(0, 0)$ — has not been turned into a general argument in this volume.

Honest conclusion of this chapter, updated: the uniqueness claim — Littrow is the only conformal retroazimuthal projection — is repeated by Snyder (1993, p. 135) and by the encyclopaedic entries on Littrow and Craig, but neither is its origin. Tracing the citation chain ([Appendix B.3](#)) leads to Maurer, *Das winkeltreue gegenazimutale Kartennetz nach Littrow* (Wiers Azimutdiagramm), *Annalen der Hydrographie und maritimen Meteorologie* 47(1–2), 1919, pp. 14–22 — a German-language paper this volume has likewise not consulted directly, so the underlying proof remains **attributed and not independently re-verified** exactly as before. What has changed is that the volume no longer misidentifies *which* inaccessible source would settle the question: Snyder’s book is a secondary report of Maurer’s result, not a competing or independent derivation of it, and obtaining *Flattening the Earth* in full would not by itself supply a proof not already absent from the volume’s other secondary sources. The claim remains plausible in light of the heuristic argument of [Proposition 13.1](#) and of the explicit, numerically verified failure of the oblique-generalisation attempt ([Section 8.3](#)), but is not proved self-sufficiently by this volume.

To Complete / Verify

Natural future work, as originally planned for this edition, now corrected in light of the above: (i) obtain Maurer (1919) — not Snyder (1993), which would not add independent content on this specific point — and report, with due attribution, its original uniqueness proof; (ii) attempt an independent proof, making explicit the retroazimuthal condition as a functional constraint on g in [Proposition 13.1](#) and checking (or disproving) its satisfiability only for C on the equator of the chosen system. Task (i) remains open (Maurer 1919 has not been obtained). Task (ii), however, has now been attempted, with a result unexpected enough to warrant its own section.

13.3 Making the heuristic rigorous: a necessary-condition analysis

Methodological Note

This section reports the outcome of actually carrying out task (ii) of the *todobox* above. The result is not the one anticipated when the task was set: rather than confirming Littrow's uniqueness, the analysis below finds no obstruction, among the necessary conditions it manages to formulate and check, to the existence of a conformal retroazimuthal projection with a *non*-equatorial centre — and an explicit numerical construction is exhibited for one such centre (the Mecca point used throughout this volume). This is reported with the same discipline as everything else in the volume: as a computed result, with its exact scope and limitations stated, not as a settled disproof of a claim found in secondary literature this author has not been able to verify against Maurer's original text. The possibility that the published claim of uniqueness rests on a requirement not captured by the analysis below — discussed in [Section 13.3.5](#) — is taken seriously and left open.

13.3.1 Step 1: the retroazimuthal condition as a Cauchy–Riemann system

Fix a centre $C = (\varphi_C, \lambda_C)$ and, following [Definition 3.2](#), seek $f = (x, y)$ with $x(P) = x(C) - r(P) \sin \theta(P)$, $y(P) = y(C) - r(P) \cos \theta(P)$, where $\theta(P) = \theta_{P \rightarrow C}$ is the known bearing function of [Eq. \(2.2\)](#) and $r(P) > 0$ is unknown. Write $s = \ln r$ and use, as coordinates, λ and the isometric latitude $\psi = \ln \tan(\pi/4 + \varphi/2)$ (so that $\sin \varphi = \tanh \psi$, $\cos \varphi = \operatorname{sech} \psi$): (λ, ψ) are the standard isothermal coordinates of the sphere, in which every orientation-preserving conformal map to the plane is, by definition, a map satisfying the ordinary planar Cauchy–Riemann equations in λ, ψ .

Proposition 13.3 (Exact differential system for r). *$f = (x, y)$ as above satisfies the Cauchy–Riemann equations $x_\lambda = y_\psi$, $x_\psi = -y_\lambda$ if and only if*

$$s_\lambda = -\theta_\psi, \quad s_\psi = \theta_\lambda. \quad (13.1)$$

Proof. Substitute $x = x_C - e^s \sin \theta$, $y = y_C - e^s \cos \theta$ into the two Cauchy–Riemann equations and expand; each becomes a linear combination of $\sin \theta$ and $\cos \theta$ with coefficients $\mp(s_\lambda + \theta_\psi)$ and $\pm(s_\psi - \theta_\lambda)$. Because $\sin \theta, \cos \theta$ are linearly independent functions (as θ is non-constant on any open set) the two coefficients must each vanish, which is [Eq. \(13.1\)](#); the converse is immediate by reversing the substitution. (Full expansion in `code/littrow_uniqueness_analysis.py`, function `s_reconstructed`, which implements exactly the system derived here, not a simplified stand-in for it.) $\square$

13.3.2 Step 2: a necessary local condition — harmonicity, for *any* centre

[Equation \(13.1\)](#) is solvable for s (locally, near any point where θ is smooth) precisely when the right-hand side is a closed differential, i.e. when $\partial_\psi(-\theta_\psi) = \partial_\lambda(\theta_\lambda)$, that is

$$\theta_{\lambda\lambda} + \theta_{\psi\psi} = 0. \quad (13.2)$$

This is exactly the heuristic content of [Proposition 13.1](#) made precise and checkable: θ must be harmonic in the isothermal coordinates.

Equation (13.2) was checked **symbolically**, not merely numerically: substituting Eq. (2.2) (with $\lambda_C = 0$ WLOG, by translation invariance in λ) into $\theta_{\lambda\lambda} + \cos \varphi \partial_\varphi(\cos \varphi \theta_\varphi)$ (the isothermal Laplacian re-expressed in φ , using $\partial_\psi = \cos \varphi \partial_\varphi$) and simplifying with SymPy, the result is **identically zero for symbolic φ_C** — not only at $\varphi_C = 0$, but for every value tried, including a fully symbolic one (code/littrow_uniqueness_analysis.py, symbolic_harmonicity_check; raw SymPy output in data/littrow_uniqueness_analysis_results.json, key step1_symbolic_harmonicity). An independent finite-difference computation, sharing no code with the symbolic one, confirms this at five test points for $\varphi_C = \text{Mecca}$ and again for $\varphi_C = 0$, with residuals of order 10^{-7} (finite-difference truncation noise, not a real discrepancy).

Remark 13.4. This is already a first correction to Proposition 13.1 as stated in Chapter 13: harmonicity of θ — the local necessary condition for a conformal retroazimuthal construction to exist — does *not* single out the equator. It holds identically, for every centre. Whatever makes Littrow’s centre special, if anything does, is not visible at this local, differential level; it can only be a *global* obstruction, on the topology of the punctured sphere, or a condition not captured by Definition 3.2 at all (Section 13.3.5).

13.3.3 Step 3: a necessary global condition — the monodromy period

The domain of θ is $\mathbb{S}^2 \setminus \{C, \bar{C}\}$, where $\bar{C}$ is the antipode of C (where the bearing toward C is undefined). Topologically this is an annulus, $\pi_1 = \mathbb{Z}$, so local solvability of Eq. (13.1) (guaranteed by Eq. (13.2)) does not by itself give a single-valued s on the whole domain: integrating $ds = -\theta_\psi d\lambda + \theta_\lambda d\psi$ around a loop generating π_1 may return a nonzero increment (a «period»), in which case $r = e^s$ would be a genuinely multi-valued, spiralling quantity, not a legitimate map.

The period was computed numerically as $\Pi(\varphi_C) = \oint \cos \varphi_1 \theta_\varphi d\lambda$ (the integral reduces to this single term on a constant- φ_1 loop, since $d\psi = 0$ there) for eight values of φ_C between 0° and 80° , including Mecca’s 21.3891° (monodromy_period_table in the same script). In every case the period is zero to within 10^{-12} — floating-point noise from a 20,000-point Simpson quadrature of a finite-difference derivative, not a genuine nonzero value. **No monodromy obstruction was found for any tested centre, equatorial or not.**

13.3.4 Step 4: explicit construction for a non-equatorial centre

Steps 2–3 remove every obstruction this analysis was able to formulate. The natural next move — absent from the original plan for this chapter, added because the negative result of Steps 2–3 demanded it — is to attempt the construction itself, for Mecca’s centre, and subject it to checks that do *not* presuppose it will work.

Concretely: $s = \ln r$ was reconstructed by numerically integrating Eq. (13.1) along an L-shaped path from a fixed reference point to each target point (function `s_reconstructed`), and the resulting $f = (x, y)$ was then tested against three properties, none of which is true *by construction* in a way that would make the test circular for (b) and (c):

- (a) **Retroazimuthal property, exactly as in Definition 3.2:** $\text{atan2}(x(C)-x(P), y(C)-y(P))$ recomputed from the constructed (x, y) and compared to $\theta_{P \rightarrow C}$ from Eq. (2.2), at six points spread over the sphere. Residual: 0.0° at all six points (this part is definitional, by the way θ enters the construction of f ; it is included as a sanity check on sign conventions, not as independent evidence).
- (b) **Cauchy–Riemann equations on (x, y) as functions of (λ, ψ) ,** evaluated by independent finite differences on the constructed numerical map — this is an independent test, since s was reconstructed via one specific numerical path integral and nothing guarantees a priori that the resulting discretised (x, y) will actually satisfy Cauchy–Riemann to good precision except if the underlying continuous construction is genuinely consistent. Maximum violation over the six test points: 3.2×10^{-6} , at finite-difference step $h = 10^{-4}$ — consistent with truncation error, not a structural failure.
- (c) **Path-independence of s :** recomputed via a second, differently-ordered L-shaped path (ψ first, then λ , against λ first, then ψ for (a)–(b)). Maximum difference over the six points: 3.3×10^{-6} , again consistent with the $n = 400$ -step trapezoidal quadrature’s truncation error, not with two genuinely different values of s at the same point.

Key Result

For $C = \text{Mecca}$ ($\varphi_C = 21.3891^\circ$, manifestly non-equatorial), an explicit numerical construction was produced that satisfies the retroazimuthal condition exactly and the Cauchy–Riemann (conformality) equations to within numerical-integration precision ($< 4 \times 10^{-6}$ on all three checks, at the tested points). Full results in `data/littrow_uniqueness_analysis_results.json` (keys `step3_mecca_*`).

13.3.5 What this section has, and has not, shown

Methodological Note

To state plainly what follows and what does not follow from the above:

What has been shown: (i) the heuristic rigidity argument of Proposition 13.1 does not, on its own, imply Littrow’s uniqueness, because its necessary condition (harmonicity of θ) holds for every centre, not just the equatorial one; (ii) a second necessary condition this analysis was able to identify — vanishing of the monodromy period, an obstruction that in principle *could* have depended on φ_C and ruled out oblique centres — also holds for every tested centre; (iii) a concrete, numerically verified (not merely formally argued) construction exists for at least one non-equatorial centre, satisfying the two defining properties (conformal, retroazimuthal) of the problem as formalised in Definition 3.2 to a precision limited only by the numerical method used, not by any residual analytic discrepancy visible in the checks performed.

What has NOT been shown: this is a single numerical construction at one centre, checked at six sample points with finite-difference and quadrature methods of finite precision, not a closed-form formula, not a proof of global injectivity (whether the resulting map is one-to-one, i.e. a genuine bijective projection and not merely a locally conformal immersion, has not been checked), not a characterisation of the map’s behaviour at or near the poles or the antipode $\bar{C}$, and not a general existence proof for *every* centre (only strong numerical evidence at nine values of φ_C for Step 3, and one explicit centre for Step 4). It is also, most importantly, **not a demonstration that the cartographic-literature claim of Littrow’s uniqueness is false:** that claim, as transmitted by the secondary sources available for this

volume, may rest on a stronger or different requirement than [Definition 3.2](#) formalises here — for instance, an implicit requirement of a closed elementary formula (as Littrow, Craig and Hammer all have, and as the construction of this section, built from a numerical path integral, explicitly does not), of global injectivity, of well-behaved regularity at both poles, or simply a different formal definition of «retroazimuthal» in Maurer’s original text than the one this volume adopted from standard modern accounts. Without access to Maurer (1919), this volume cannot adjudicate between «the literature’s uniqueness claim is about a strictly stronger property» and «the literature’s uniqueness claim is simply imprecise as usually transmitted». Both remain open possibilities.

13.4 Injectivity and boundary behaviour of the Mecca construction

This section takes up item (ii) of the *todobox* at the end of [Section 13.3.5](#) — global injectivity of the numerical construction of [Section 13.3](#), and its behaviour at the poles and at $\bar{C}$ — the one item of the three originally listed there that does not require access to Maurer (1919). Item (i) (a closed-form expression at an oblique centre) and item (iii) (Maurer 1919) remain exactly as open as before; they are not addressed here. All computations below reuse, without modification, the functions `bearing`, `psi_of_phi`, `phi_of_psi` and `f_map` of `code/littrow_uniqueness_analysis.py` ([Section 13.3](#)); the new code, in `code/mecca_injectivity_poles.py`, only combines them differently.

13.4.1 Global injectivity: a sampled search for self-intersections

A grid of 358 points was built over $\varphi \in [-70^\circ, 70^\circ]$ (step 10°) and $\lambda \in [-180^\circ, 180^\circ]$ (step 15°), excluding an 8° neighbourhood of C (Mecca) and of $\bar{C}$ where the construction is either singular or numerically unreliable ([Section 13.4.3](#) below). For each grid point the image $(x, y) = f(\varphi, \lambda)$ was computed, and every pair of points was classified as «local» (angular separation on the sphere below 25°) or «non-local» (above 25°); the minimum planar image distance was recorded separately within each class.

Key Result

Minimum image-plane distance between a *non-local* pair (angular separation $\geq 25^\circ$): 0.0311, attained between $(\varphi, \lambda) = (0^\circ, -75^\circ)$ and $(0^\circ, -30^\circ)$, 45.0° apart on the sphere. Minimum image-plane distance between a *local* pair (angular separation $< 25^\circ$, the natural comparison scale): 0.0106, between two grid points only 15° apart. The overall diagonal of the sampled image (max – min in x and y , combined) is 7.89. No non-local pair was found closer in the image than the closest local pair: on this 358-point sample, distant points on the sphere do not fold onto nearby image points. Full results in `data/mecca_injectivity_poles_results.json`, key `injectivity_scan`.

Methodological Note

This is a sampled numerical search, not a proof: it shows no evidence of self-intersection on the 358 tested points and the pairs among them, at the resolution of a $10^\circ \times 15^\circ$ grid excluding 8° around C and $\bar{C}$. A genuine but localised fold smaller than the grid spacing, or one confined to the excluded neighbourhoods, would not be detected by this test. Together with the already-established fact that the map is a local diffeomorphism wherever $s = \ln r$ is smooth (its Jacobian equals $e^{2s} > 0$ identically, by conformality, so it never

vanishes away from $C, \bar{C}$), this raises the balance of evidence towards global injectivity on the tested region without settling the question in general.

13.4.2 Behaviour at the poles: an unexpected second singularity

The construction of Section 13.3 was built to handle the two points where the bearing function $\theta_{P \rightarrow C}$ itself is singular, C and $\bar{C}$; nothing in its derivation suggested that the geographic poles — ordinary points for θ , where the bearing towards Mecca is simply due south (north pole) or involves no singularity at all — should behave specially. They do.

Points near the north pole ($\varphi = 89^\circ$) at six longitudes $0^\circ, 60^\circ, \dots, 300^\circ$ were mapped and their images compared:

Key Result

At $\varphi = 89^\circ$, the six images have radius $r = |f(P)|$ ranging only from 56.21 to 56.94 (a 1.3% spread, consistent with finite-difference and quadrature noise) while their angular position $\arg f(P)$ in the plane tracks the longitude directly, advancing by approximately 60° per 60° of λ (measured values: $-40.11^\circ, 20.28^\circ, 80.53^\circ, 140.39^\circ, 199.99^\circ, 259.99^\circ$, i.e. six points $\approx 60^\circ$ apart, matching the six input longitudes up to an overall rotation). The same qualitative pattern was found independently at the south pole ($\varphi = -89^\circ$): matching radii to four significant figures, longitude-dependence identical in form.

In other words, the pole is not sent to a single point, as it is for every classical projection used elsewhere in this volume (Craig, Hammer): each pole is approached along what is, in the limit, a circle of finite radius on which the angular coordinate simply re-encodes the longitude of approach. A single point on the sphere (the pole) does not have a single image; the limit depends on the direction of approach. This was checked to be a genuine trend, not a plateau, by tracking the radius at five successive latitudes on the way to the pole, at fixed $\lambda = 0^\circ$:

φ	$\psi = \ln \tan(\pi/4 + \varphi/2)$	r	$\ln r/\psi$
70°	1.735	2.51	0.530
85°	3.131	11.01	0.766
89°	4.741	56.28	0.850
89.5°	5.434	112.86	0.870
89.8°	6.351	282.60	0.889
89.95°	7.737	1131.28	0.909

Methodological Note

The radius grows without bound as $\varphi \rightarrow 90^\circ$, and the ratio $\ln r/\psi$ climbs steadily towards a value close to 1, consistent with r growing asymptotically like $e^\psi = \tan(\pi/4 + \varphi/2)$, i.e. with $s_\psi = \partial(\ln r)/\partial\psi \rightarrow 1$ at the pole — a clean, if not fully asymptotically proved, characterisation of the divergence rate, not merely the qualitative observation that it diverges. This was checked along $\lambda = 0^\circ$ only; the six-longitude snapshot above establishes that the *radius* of divergence does not depend on λ , only the angular position on the limiting circle does, so the rate found here should be representative of every longitude.

Both poles, previously unremarked in the construction, are therefore a *second* kind of singular locus for this conformal retroazimuthal map, distinct in nature from C and $\bar{C}$ (where θ itself

is discontinuous): here θ is perfectly well-behaved, but the reconstructed radial coordinate r diverges directionally, so that the map fails to extend continuously to the poles at all, let alone injectively. None of the three historical constructions of this volume (Littrow, Craig, Hammer) has this defect at the poles; whether it is a necessary feature of *any* conformal retroazimuthal construction with a non-equatorial centre, or an artefact specific to the particular numerical path-integral construction of Section 13.3, is left open.

13.4.3 Behaviour near $\bar{C}$

Eight points at each of five decreasing angular distances ($10^\circ, 5^\circ, 2^\circ, 1^\circ, 0.5^\circ$) from $\bar{C}$ were generated by direct spherical navigation along eight initial bearings ($0^\circ, 45^\circ, \dots, 315^\circ$ measured at $\bar{C}$), and mapped.

Key Result

Along seven of the eight tested directions (45° through 315°), the radius shrinks cleanly and consistently as the point approaches $\bar{C}$: at $\delta = 1^\circ$, $r \approx 0.0198\text{--}0.0200$ across all seven; at $\delta = 0.5^\circ$, $r \approx 0.0099\text{--}0.0100$ — almost exactly halved, consistent with $r \propto \delta$ near $\bar{C}$ along these directions, and with negligible dependence on the specific direction among the seven. The eighth direction (0° , due north from $\bar{C}$, along its own meridian) is a clear outlier: $r = 1.011$ at $\delta = 1^\circ$ and $r = 217.3$ at $\delta = 0.5^\circ$ — two orders of magnitude larger than the other seven directions at the same distance, and *growing*, not shrinking, as δ decreases.

The seven-direction pattern is exactly what the volume's own account of $\bar{C}$ predicts: $\theta_{P \rightarrow C}$ is discontinuous at $\bar{C}$ (Chapter 13), so a direction-dependent limit for r approaching it is the expected signature of that discontinuity, not a numerical defect. The eighth direction's outlying behaviour, however, is now diagnosed rather than merely flagged, as follows.

Diagnosis: a crossing artefact of the L -shaped path, not a genuine limit

The construction of Section 13.3 reconstructs $s = \ln r$ by integrating along an L -shaped path from a fixed reference point $(\lambda_0, \varphi_0) = (-90^\circ, -40^\circ)$ to the target: first a longitude leg at fixed latitude φ_0 , then a latitude leg at fixed longitude λ_{target} . For the eighth direction (bearing 0° , due north from $\bar{C}$), spherical navigation leaves the longitude unchanged, so $\lambda_{\text{target}} = \lambda_{\bar{C}}$ exactly; the second leg of the path is then a vertical segment, at $\lambda = \lambda_{\bar{C}}$, running from $\varphi_0 = -40^\circ$ up to a target just north of $\varphi_{\bar{C}} = -21.3891^\circ$ — and therefore passes *directly through* $\bar{C}$ itself, the one point where $\theta_{P \rightarrow C}$ is discontinuous by definition. No other direction among the eight has this property at this reference point: direction 180° (due south) stays on the same meridian but moves *away* from φ_0 towards more negative latitudes, never crossing $\varphi_{\bar{C}}$, and the remaining six directions leave $\lambda_{\bar{C}}$ altogether. This is a property of the specific reference point and path convention, not of the point $\bar{C}$ or the direction 0° in themselves.

Key Result

Recomputing r along all eight directions with the integration order reversed (latitude leg first, at fixed $\lambda_0 = -90^\circ$, then longitude leg at fixed φ_{target} — a path whose fixed-longitude leg is at λ_0 , nowhere near $\lambda_{\bar{C}} = -140.14^\circ$) makes the eighth direction fall directly into line with the other seven: at $\delta = 1^\circ$, $r = 0.01973$ (versus $0.0198\text{--}0.0200$ for the other seven under the original path); at $\delta = 0.5^\circ$, $r = 0.009883$ (versus $0.0099\text{--}0.0100$). The two path orders agree to four or more significant figures on the seven previously well-behaved directions (e.g. direction 45° at $\delta = 1^\circ$: 0.0197835 vs. 0.0197835), confirming they

are computing the same quantity away from any crossing. Symmetrically, the reversed path introduces its own crossing artefact at direction 270° (whose longitude leg, at fixed $\varphi \approx \varphi_{\bar{C}}$, sweeps across $\lambda_{\bar{C}}$ for this direction and this reference point), while the original path remains clean there. Full results, both path orders, all eight directions and five distances: `data/cbar_anomaly_diagnosis_results.json`.

Methodological Note

The eighth-direction anomaly is therefore a crossing artefact of the particular L -path and reference point used to reconstruct s , not a genuine feature of the limit of r as $P \rightarrow \bar{C}$: whichever of the two path orders happens not to cross $\bar{C}$ for a given direction recovers the same clean, direction-independent-up-to-the-expected-scatter pattern ($r \propto \delta$, magnitude $0.0197\text{--}0.0200$ at $\delta = 1^\circ$) found on the other seven directions. This resolves the item cleanly in one direction (crossing artefact, not a new limit) without requiring any claim about *which* path order is «correct» in general — neither is: each has exactly one direction, for this reference point, along which it is unusable, and the fix used here (taking, for each direction, the path order that does not cross $\bar{C}$) is a diagnostic device for this section, not a general prescription adopted elsewhere in the volume.

As a by-product, the same two-path comparison was applied to the pole singularity of [Section 13.4.2](#): at every tested latitude and longitude near both poles, the two integration orders agree to 4–5 significant figures (maximum relative difference 2.3×10^{-5} across all points tested, consistent with finite-difference and quadrature noise rather than a systematic effect). Unlike the $\bar{C}$ anomaly just diagnosed, the pole behaviour is therefore *not* a crossing artefact of this particular path convention: it persists identically under a differently-ordered integration path. This does not by itself prove the pole singularity is a necessary feature of *any* conformal retroazimuthal construction with a non-equatorial centre (a question about the mathematical object itself, not about this or that way of computing it) — but it does rule out, specifically, the explanation that would have been most readily available (a path-order artefact analogous to the one just found at $\bar{C}$), and so shifts the item from «undiagnosed» to «genuinely open at the level of the construction itself».

To Complete / Verify

Concretely open after this section: (i) obtain a closed-form expression for $r(\varphi, \lambda)$ at an oblique centre, if one exists, rather than the numerical path-integral construction given here; (ii) settle whether the pole singularity of [Section 13.4.2](#) is intrinsic to any oblique-centre conformal retroazimuthal construction, or an artefact of the path-integral *method of construction* itself (now checked, and found robust, against the one specific artefact — integration-path crossing — diagnosed in this section for $\bar{C}$; not checked against other possible sources of method-specific artefact); (iii) still, as before, obtain Maurer (1919) to settle what the original uniqueness claim actually asserts and on what grounds, and confirm whether Maurer documents anything resembling the pole behaviour found here. Global injectivity ([Section 13.4](#)) now has sampled numerical evidence in its favour though not a proof; the eighth-direction anomaly at $\bar{C}$, raised as newly open earlier in this section, is now resolved (crossing artefact, diagnosed and confirmed above) and is removed from this list. Item (i) is taken up, and resolved, in [Section 13.5](#) below.

13.5 A closed-form solution via complex analysis

Methodological Note

This section takes up item (i) of the *todobox* above: whether the numerical path-integral construction of Section 13.3 can be replaced by an elementary closed-form expression. Unlike the exploratory numerical work of the rest of this chapter, the argument below is a derivation: each step is a symbolic identity, checked by independent computer algebra, not a fit to numerical output. It is reported with the same discipline as elsewhere: what is proved symbolically is distinguished from what is only checked numerically, and the scope of the result — what it does and does not settle about the uniqueness question of Chapter 13 — is stated explicitly at the end.

13.5.1 An auxiliary holomorphic function

Fix, as before, $\lambda_C = 0$ WLOG and write $\zeta = \lambda + i\psi$ for the isothermal complex coordinate. Define the auxiliary function

$$g(\zeta) = \sin \varphi_C + i \cos \varphi_C \sin \zeta, \quad (13.3)$$

manifestly holomorphic in ζ (a sum of a constant and $i \cos \varphi_C$ times the holomorphic function $\sin \zeta$).

Proposition 13.5 (Bearing as the argument of g). *With N, D the two arguments of Eq. (2.2) (so that $\theta_{P \rightarrow C} = \text{atan2}(N, D)$),*

$$D(\varphi, \lambda) - iN(\varphi, \lambda) = \cos \varphi g(\zeta) \quad (13.4)$$

identically, for every φ_C (not only Mecca's).

Proof. Substitute $\sin \varphi = \tanh \psi$, $\cos \varphi = \text{sech } \psi$ into $D - iN$ and expand $\sin \zeta = \sin \lambda \cosh \psi + i \cos \lambda \sinh \psi$ in $g(\zeta)$; both sides reduce, after using $1 \pm \sin \varphi = \cos \varphi e^{\pm \psi}$ and collecting terms in $e^{i\lambda - \psi}$ and $e^{-i\lambda + \psi}$, to the same expression. Checked by SymPy for fully symbolic φ_C (code/oblique_closed_form.py, symbolic_identity_check: the simplified difference of the two sides is the literal expression 0, not a numerically small residual) and confirmed independently at 2000 random points $(\lambda, \psi, \varphi_C)$, maximum floating-point discrepancy 2.5×10^{-16} (machine epsilon). $\square$

Since $\cos \varphi > 0$ away from the poles, Eq. (13.4) gives $e^{-i\theta_{P \rightarrow C}} = (D - iN)/\sqrt{N^2 + D^2} = g(\zeta)/|g(\zeta)|$, i.e. $\theta_{P \rightarrow C} = -\arg g(\zeta)$ exactly: **the bearing toward an arbitrary centre C is, up to sign, the argument of a single holomorphic function of the isothermal coordinate.**

13.5.2 From holomorphy of g to a closed form for r

Away from the zeros of g (which, by direct check, occur exactly at ζ corresponding to C and $\bar{C}$ — consistent with these being precisely the two points excluded from the domain of θ), $\ln g(\zeta) = \ln |g(\zeta)| + i \arg g(\zeta)$ is holomorphic, being the composition of the holomorphic logarithm with the holomorphic non-vanishing g . Its real and imaginary parts are therefore, by the most elementary fact of the subject, harmonic conjugates: they satisfy the planar Cauchy–Riemann system in (λ, ψ) automatically, with no separate integration required.

Key Result

Setting $s = \ln |g(\zeta)|$ and comparing with $\theta = -\arg g(\zeta)$ established above, $(s, -\theta) = (\operatorname{Re} \ln g, \operatorname{Im} \ln g)$ satisfies Eq. (13.1) of Proposition 13.3 **identically**, checked symbolically for symbolic φ_C (`symbolic_harmonic_conjugate_check`; both CR residuals simplify to 0). This is exactly the relation the chapter's own derivation calls for — $s_\lambda = -\theta_\psi$, $s_\psi = \theta_\lambda$ — verified here not by numerical differentiation of a path-integral reconstruction, as in Section 13.3, but as an algebraic consequence of $\ln g$ being holomorphic.

Because $r = e^s$ is determined by s only up to the additive real constant that $\ln g$ leaves free (any two holomorphic functions with the same imaginary part differ by a real constant), the natural normalisation $r = |g(\zeta)|$ (constant = 1) is adopted; this is the same normalisation, verified below, that reproduces Littrow's own formula exactly at $\varphi_C = 0$, and is retained as the canonical choice throughout this section.

13.5.3 The closed form, in full

Writing out $\operatorname{Im}(-i g(\zeta))$ and $\operatorname{Re}(-i g(\zeta))$ explicitly (using $f - f_C = -i e^{\ln g(\zeta)} = -i g(\zeta)$, the same relation between f and g used implicitly throughout) gives, with λ already relative to C 's meridian:

$$x(\varphi, \lambda) = \frac{\sin \lambda \cos \varphi_C}{\cos \varphi}, \quad y(\varphi, \lambda) = \tan \varphi \cos \varphi_C \cos \lambda - \sin \varphi_C \quad (13.5)$$

Methodological Note

Equation (13.5) contains no numerical integration, no path convention, and no reference point: it is an elementary formula in $\varphi, \lambda, \varphi_C$, exactly on the same footing as Eq. (8.1) (Littrow), Eq. (9.1) (Craig) or Eq. (10.1) (Hammer). Setting $\varphi_C = 0$ collapses it, term by term, to Eq. (8.1): checked symbolically (`reduces_to_littrow_check`), both differences simplify to 0.

13.5.4 Independent verification**Key Result**

Retroazimuthal property, proved symbolically for symbolic φ_C , not merely checked pointwise: writing $(-x, -y) \cdot (D, N)^\perp$ -type cross and dot products between the image vector and (N, D) , the cross product $(-x)D - (-y)N$ simplifies identically to 0 and the dot product $(-x)N + (-y)D$ simplifies to $(N^2 + D^2)/\cos \varphi > 0$ (`code/oblique_closed_form.py`): the image vector from $f(P)$ to $f(C)$ is, for every $\varphi, \lambda, \varphi_C$ where $\cos \varphi \neq 0$, exactly parallel to and codirectional with (N, D) , hence $\operatorname{atan2}(x(C)-x(P), y(C)-y(P)) = \theta_{P \rightarrow C}$ exactly, not to within numerical tolerance.

Conformality, proved symbolically: the Cauchy–Riemann equations for (x, y) as functions of (λ, ψ) , evaluated directly by symbolic differentiation of Eq. (13.5) (rewritten with $\tan \varphi = \sinh \psi$, $1/\cos \varphi = \cosh \psi$), both simplify to 0 for symbolic φ_C (`symbolic_cr_check`).

Large-scale independent numerical check: 20,000 pairs (C, P) with both φ_C and (φ, λ) drawn uniformly at random over the sphere (excluding an 8° neighbourhood of $C, \bar{C}$ and the immediate vicinity of the poles), evaluated by a freshly-written, independent script

sharing no code with [Section 13.3](#): maximum bearing error 1.7×10^{-12} degrees — floating-point noise, not a residual discrepancy, and over two orders of magnitude smaller than the 252-point native-Littrow check of [Chapter 8](#) because no finite-difference or quadrature step is involved anywhere in this construction.

Consistency with the earlier path-integral construction: at the six test points of [Section 13.3](#), for Mecca’s centre, r from [Eq. \(13.5\)](#) matches $r = e^s$ from the numerical path integral to 4–5 significant figures *after* accounting for the one free real constant the two constructions fix differently (the path integral sets $s = 0$ at an arbitrary reference point; the closed form sets the constant via $r = |g(\zeta)|$): maximum relative error 2.1×10^{-6} , consistent with the path integral’s own reported truncation precision ($< 4 \times 10^{-6}$, [Section 13.3](#)) and not with a genuine discrepancy between the two constructions. Full data in `data/oblique_closed_form_results.json`.

13.5.5 What this does, and does not, settle

Methodological Note

What has been shown: a closed-form, elementary (finitely many elementary functions, no integrals, no series) conformal retroazimuthal projection exists for *every* centre φ_C , not only the equatorial one — resolving item (i) of the previous *todo*box in the affirmative, and doing so by proof (symbolic identities, verified independently by computer algebra) rather than by the numerical construction of [Section 13.3](#), which this section supersedes as the primary evidence for existence, while remaining consistent with it to numerical precision.

What has NOT been shown: global injectivity of [Eq. \(13.5\)](#) away from the sampled region of [Section 13.4](#) (not re-examined here; the closed form and the earlier numerical construction are the same map up to a normalisation constant, so the earlier sampled evidence transfers, but remains sampled evidence, not a proof); good behaviour at the poles — on the contrary, [Eq. \(13.5\)](#) makes manifest, and therefore *explains* rather than resolves, the pole singularity of [Section 13.4.2](#): $\cos \varphi \rightarrow 0$ in the denominator of x is a structural feature of the closed form itself, present already in Littrow’s own $\varphi_C = 0$ case (where it is a known, longstanding property of Littrow, not a new finding), and it is now clear that the oblique case inherits this defect from Littrow rather than introducing a new one specific to oblique centres. **Most importantly:** this result sharpens, rather than resolves, the tension with the literature flagged in [Chapter 13](#). If the cartographic literature’s claim that Littrow is the *only* conformal retroazimuthal projection is to be reconciled with [Eq. \(13.5\)](#), the reconciliation must rest on a requirement not captured by [Definition 3.2](#) — global injectivity, regular behaviour at the poles, or a different formal definition altogether (as already flagged in [Section 13.3.5](#)) — since existence of an elementary closed form, the one gap that construction had left open, is now closed. Without Maurer (1919), this volume still cannot adjudicate which of these it is; what has changed is that the burden on that reconciliation is now heavier, since the objection «no closed form is known for oblique centres» no longer applies.

To Complete / Verify

Concretely open after this section: (i) global injectivity of [Eq. \(13.5\)](#) as a genuine proof, not sampled evidence (unchanged from [Section 13.4](#)); (ii) Maurer (1919), to determine whether its uniqueness claim uses a stronger notion of «retroazimuthal projection» than [Definition 3.2](#), and whether it already contains [Eq. \(13.5\)](#) or an equivalent under a differ-

ent name; (iii) a characterisation, analogous to Eq. (13.5), of the constructions' behaviour under composition or comparison with Craig and Hammer at the same oblique centre, not attempted in this volume.

13.6 Resolving item (i): global injectivity fails, exactly and everywhere, by an explicit fold

Methodological Note

This section takes up item (i) of the *todobox* above. It was undertaken expecting, at best, further sampled evidence of the kind already gathered in Section 13.4. The outcome is instead a closed-form answer, of the opposite sign from what the chapter's own heading once anticipated: Eq. (13.5) is **not** globally injective, for *any* centre φ_C — not merely at Mecca, and not merely away from some numerically inaccessible region. The failure is exact, provable by direct substitution, and — as Section 13.6.4 argues — turns out to reconcile rather than deepen the tension with the cartographic literature flagged throughout this chapter.

13.6.1 The map is a scaled sine function of the isothermal coordinate

Recall from Section 13.3 that, writing $\zeta = \lambda + i\psi$ for the isothermal coordinate (with λ already relative to C 's meridian) and $a = \varphi_C$, the closed form of Eq. (13.5) satisfies $x + iy = -i g(\zeta)$ with $g(\zeta) = \sin a + i \cos a \sin \zeta$ (`code/oblique_closed_form.py`, already verified there symbolically). A direct expansion gives

$$x(\varphi, \lambda) + i y(\varphi, \lambda) = \cos \varphi_C \sin \zeta - i \sin \varphi_C, \quad (13.6)$$

i.e. the closed-form projection is, up to a rigid vertical translation of the image plane by the constant $-i \sin \varphi_C$ (which does not affect injectivity) and a positive real scale factor $\cos \varphi_C$ (same proviso), **exactly the complex sine function of the isothermal coordinate** $\zeta = \lambda + i\psi$. Injectivity of the retroazimuthal map is therefore, without further approximation, injectivity of $\zeta \mapsto \sin \zeta$ on the domain $\lambda \in (-\pi, \pi]$ (a full period, since λ is a genuine angular coordinate), $\psi \in \mathbb{R}$.

13.6.2 The classical non-injectivity of a full period of sin, transplanted to the sphere

Proposition 13.6 (Exact fold identity). *For every $\varphi, \lambda, \varphi_C$ with $\cos \varphi \neq 0$, $\cos \varphi' \neq 0$, where $\varphi' = -\varphi$ and $\lambda' = \pi - \lambda$ (relative to C 's meridian), the closed form of Eq. (13.5) satisfies*

$$x(\varphi', \lambda') = x(\varphi, \lambda), \quad y(\varphi', \lambda') = y(\varphi, \lambda).$$

Proof. This is the classical identity $\sin(\pi - \zeta) = \sin \zeta$, valid for every complex ζ , applied to $\zeta' = \pi - \lambda - i\psi = \pi - \zeta$ (using $\psi(-\varphi) = -\psi(\varphi)$, since the isometric latitude $\psi = \ln \tan(\pi/4 + \varphi/2)$ is an odd function of φ , itself an elementary check: $\tan(\pi/4 - \varphi/2) = 1/\tan(\pi/4 + \varphi/2)$). Rather than invoke this shortcut, the identity was re-derived independently, directly on Eq. (13.5) in real coordinates (φ, λ) , without reference to g or ζ at all: symbolic substitution of $(-\varphi, \pi - \lambda)$ into Eq. (13.5) and expansion in elementary trigonometric identities

(code/global_injectivity_folding.py, function `symbolic_fold_identity`) gives a difference that simplifies identically to 0 for both x and y , for symbolic $\varphi, \lambda, \varphi_C$. $\square$

Key Result

The involution $\iota(\varphi, \lambda) = (-\varphi, \pi - \lambda)$ (longitude measured from C 's meridian) is an exact symmetry of the closed-form projection, for every centre φ_C , proved symbolically, not sampled. Numerically, at both Mecca's centre and the native equatorial (Littrow) centre, 5000 random points P and their images $\iota(P)$ produce identical (x, y) to within 1.5×10^{-14} — floating-point noise, confirming the symbolic derivation rather than adding to it. Full data in `data/global_injectivity_folding_results.json`, key `claim3_numerical_fold_check`.

This holds for $\varphi_C = 0$ as well: **Littrow's own native projection is subject to exactly the same fold.** Nothing in Chapter 8 claimed otherwise — the point was simply never raised there, and the 252-point verification of Chapter 8 tested the retroazimuthal property, not injectivity. This is not a defect introduced by the oblique generalisation of this chapter; it is a property of the retroazimuthal construction itself, present from 1833 onward, that this volume had not previously had the means to state in closed form.

13.6.3 The fold is a genuine identification of distinct points, not a coordinate artefact

Methodological Note

Two clarifications are needed before treating Proposition 13.6 as a substantive finding rather than a labelling accident.

First, ι has exactly two fixed points on the domain where $\cos \varphi \neq 0$: solving $\varphi = -\varphi$ forces $\varphi = 0$, and then $\lambda = \pi - \lambda \pmod{2\pi}$ forces $\lambda = \pi/2$ or $\lambda = 3\pi/2 \equiv -\pi/2$ (relative to C 's meridian) — solved symbolically (code/global_injectivity_folding.py, function `symbolic_fixed_points_and_branch_points`). These two points are exactly the zeros of $g'(\zeta) = i \cos \varphi_C \cos \zeta$ (again solved symbolically, same function), i.e. the two branch points of the map, where the local conformal-diffeomorphism property established in Section 13.4 necessarily breaks down. Away from these two points, ι moves every point to a genuinely different one.

Second, this is confirmed to be a real geographic identification, not a relabelling of the same point: over 2000 random test points at Mecca's centre, the angular separation on the sphere between P and $\iota(P)$ is never below 6.1° in the sample (the minimum found, not a bound; points arbitrarily close to a branch point would show arbitrarily small separations, but generic points do not). A concrete example: $(\varphi, \lambda) = (-2.93^\circ, 129.04^\circ)$ and its fold partner $(2.93^\circ, 130.67^\circ)$, 6.08° apart on the sphere, map to the identical point of the plane. Full data in `data/global_injectivity_folding_results.json`, key `claim4_fold_pairs_distinct`.

13.6.4 The natural fundamental domain: a lune of longitudinal half-width 90°

Methodological Note

A map that is exactly 2-to-1, branched at two points, restricts to a bijection on any fundamental domain of ι that excludes exactly one point of each fold pair. The classical fact about $\sin$ on a full period strip — that it restricts to an injective (indeed conformal bijective) map on the half-period open strip $\{\zeta : -\pi/2 < \operatorname{Re} \zeta < \pi/2\}$ — transplants directly: in geographic terms, the candidate fundamental domain is the open **lune** of longitudinal half-width 90° centred on C 's meridian, $|\lambda - \lambda_C| < 90^\circ$, covering *every* latitude (not a hemisphere in the usual sense bounded by a great circle perpendicular to the $C-\bar{C}$ axis, but a lune bounded by two meridians 180° apart).

Key Result

Sampled confirmation, Mecca's centre: 20,000 points drawn uniformly at random inside the open lune $\lambda \in (\lambda_C - 90^\circ, \lambda_C + 90^\circ) = (-50.14^\circ\text{E}, 129.86^\circ\text{E})$, $\varphi \in (-85^\circ, 85^\circ)$, checked pairwise (nearest-neighbour proxy in the image plane, same sampled-search method as [Section 13.4](#)): minimum image distance between distinct sampled points 3.18×10^{-5} , consistent with sampling density (nearest sampled points are themselves only $\sim 0.03^\circ$ apart in λ) and giving no indication of a collision. This is, as throughout this volume, sampled evidence *consistent with* injectivity on the lune, not a proof of it: a genuine proof would require the classical strip-injectivity fact for $\sin$ to be re-derived here rather than invoked, which has not been done. Full data in `data/global_injectivity_folding_results.json`, key `claim5_lune_injectivity_sample`.

What this changes in the chapter's overall conclusion. The tension flagged in [Chapter 13](#) and sharpened at the end of the previous section — between the cartographic literature's claim that Littrow is the unique conformal retroazimuthal projection, and the existence of an elementary closed form ([Eq. \(13.5\)](#)) for every centre — is not deepened by this section; it is **plausibly reconciled**. If «a conformal retroazimuthal projection», as intended by the literature, is implicitly understood to mean an injective one (a genuine map of a usable portion of the globe, not merely a locally-conformal immersion), then [Eq. \(13.5\)](#) does not contradict uniqueness at all: it is a map that fails to be a bona-fide projection on the whole sphere for *every* centre, Littrow's included, and is only ever used, in practice, on a restricted portion of the globe — which is consistent with the practical use of Littrow-type qibla charts described in [Chapter 4](#), typically presented for a limited region around the point of interest rather than for the whole world. This volume does not claim to have found the reconciliation Maurer (1919) intends (that requires the source itself, still unobtained, cf. item (ii) of the *todobox* below); it claims only to have found *a* reconciliation consistent with everything established in this volume, resting on a closed-form, symbolically verified obstruction rather than on a sampled absence of evidence.

13.7 Resolving item (i) analytically: sin is injective on the open half-period strip

Methodological Note

Section 13.6.4 invoked, without re-deriving, the classical fact that sin restricts to an injective map on the open strip $\{\zeta : -\pi/2 < \operatorname{Re} \zeta < \pi/2\}$, and reported only sampled evidence consistent with it. This section replaces the sampled evidence with a self-contained proof, using nothing beyond the complex addition formulas for sin and cos (themselves consequences of Euler's formula) and elementary algebra — not by citing a textbook statement of strip-injectivity as already established.

Lemma 13.7 (Zero sets of complex sin and cos). *For $z \in \mathbb{C}$: $\cos z = 0$ if and only if $z = \pi/2 + k\pi$ for some integer k ; $\sin z = 0$ if and only if $z = k\pi$ for some integer k . In particular, both zero sets are purely real.*

Proof. Writing $\cos z = (e^{iz} + e^{-iz})/2$, $\cos z = 0$ is equivalent to $e^{2iz} = -1 = e^{i\pi}$, i.e. $2iz = i\pi + 2ik\pi$ for some integer k (the complex exponential is $2\pi i$ -periodic and injective on any period strip of its argument), i.e. $z = \pi/2 + k\pi$, a real number. The argument for $\sin z = (e^{iz} - e^{-iz})/(2i)$ is identical, using $e^{2iz} = 1$ in place of $e^{2iz} = -1$, giving $z = k\pi$. (Verified independently, symbolically, in `code/sin_strip_injectivity_check.py`, functions `check_zero_locus_cos` and `check_zero_locus_sin`.) $\square$

Proposition 13.8 (Injectivity of sin on the open half-period strip). *Let $S = \{\zeta \in \mathbb{C} : -\pi/2 < \operatorname{Re} \zeta < \pi/2\}$. If $\zeta_1, \zeta_2 \in S$ and $\sin \zeta_1 = \sin \zeta_2$, then $\zeta_1 = \zeta_2$.*

Proof. The prosthaphaeresis identity $\sin \zeta_1 - \sin \zeta_2 = 2 \cos\left(\frac{\zeta_1 + \zeta_2}{2}\right) \sin\left(\frac{\zeta_1 - \zeta_2}{2}\right)$, an algebraic consequence of the complex addition formulas for sin and valid for every $\zeta_1, \zeta_2 \in \mathbb{C}$ (verified symbolically in `code/sin_strip_injectivity_check.py`, function `check_prosthaphaeresis_identity`), shows that $\sin \zeta_1 = \sin \zeta_2$ forces

$$\cos\left(\frac{\zeta_1 + \zeta_2}{2}\right) = 0 \quad \text{or} \quad \sin\left(\frac{\zeta_1 - \zeta_2}{2}\right) = 0. \quad (13.7)$$

By Lemma 13.7, the first alternative forces $(\zeta_1 + \zeta_2)/2 = \pi/2 + k\pi$ for some integer k , i.e. $\zeta_1 + \zeta_2 = \pi + 2k\pi$, a real number; writing $\zeta_j = x_j + iy_j$, this means $y_1 + y_2 = 0$ and $x_1 + x_2 = \pi + 2k\pi$. But $x_1, x_2 \in (-\pi/2, \pi/2)$ gives $x_1 + x_2 \in (-\pi, \pi)$ strictly, so no integer k makes $x_1 + x_2 = \pi + 2k\pi$ solvable: the first alternative is impossible for $\zeta_1, \zeta_2 \in S$. The second alternative, again by Lemma 13.7, forces $(\zeta_1 - \zeta_2)/2 = k\pi$, i.e. $\zeta_1 - \zeta_2 = 2k\pi$, real; so $y_1 = y_2$ and $x_1 - x_2 = 2k\pi$, and $x_1, x_2 \in (-\pi/2, \pi/2)$ gives $x_1 - x_2 \in (-\pi, \pi)$ strictly, forcing $k = 0$ and hence $x_1 = x_2$. Combined with $y_1 = y_2$, this gives $\zeta_1 = \zeta_2$. $\square$

Key Result

Item (i) of the tobox above is resolved analytically, not merely sampled. Proposition 13.8 is exactly the classical strip-injectivity fact for sin that Section 13.6.4 had invoked without proof; combined with Eq. (13.6) (the closed-form retroazimuthal map is, up to a translation and a positive real scale, the map $\zeta = \lambda + i\psi \mapsto \sin \zeta$) and with λ ranging over a full period, this shows rigorously that the open lune $|\lambda - \lambda_C| < 90^\circ$ is **exactly a fundamental domain of ι** : the closed-form retroazimuthal construction, for every centre φ_C , restricts to a genuinely injective map on it. The 20,000-point sampled check of Sec-

tion 13.6.4 is thus superseded, on this specific point, by a proof; the sampled check itself is left in the text as independent numerical corroboration, in keeping with this volume's practice of not deleting evidence once a stronger result subsumes it. An independent numerical scan of 20,000 additional random points across the whole strip (not just at Mecca's or Littrow's centre, since [Proposition 13.8](#) is a statement about $\sin$ itself, centre-independent) found no image-plane collision at all, consistent with, and not required by, the proof (code/sin_strip_injectivity_check.py, key numerical_collision_scan; full data in data/sin_strip_injectivity_results.json).

Methodological Note

What this does *not* do: it does not touch item (iv) below (Craig and Hammer, whose formulas are not scaled complex sines of an isothermal coordinate, so [Proposition 13.8](#) does not apply to them directly); nor does it settle whether the closed boundary of the lune, $|\lambda - \lambda_C| = 90^\circ$ (where the two branch points of [Proposition 13.6](#) sit), can be included on one side to obtain a maximal domain strictly larger than the open lune — a question about the boundary, not the interior, left unexamined here since it was not part of item (i) as originally stated.

To Complete / Verify

Concretely open after this section: (i) [**resolved above**: the lune is proved, not merely sampled, to be a fundamental domain of ι]; (ii) [**attempted, not resolved**: a search was carried out for Maurer (1919) itself, see [Section 13.8](#) below — the original paper remains unobtained, though its likely digitisation venue has now been narrowed down]; (iii) [**resolved below**: yes, an equivalent restriction is documented in the literature, under the description that retroazimuthal world maps «overlap themselves»]; (iv) [**resolved for Craig, open for Hammer**: see [Section 13.8](#) below for a closed-form fold for Craig, exactly analogous to ι ; no comparable closed form has been found for Hammer, only sampled evidence consistent with front-hemisphere injectivity].

13.8 Resolving items (iii) and (iv)

13.8.1 Item (iii): the fold is already named in the qibla literature, as an overlap

Methodological Note

A search was carried out (not merely an internal cross-check of [Chapter 4](#)) for whether the fold identified in [Proposition 13.6](#) appears, under any name, in the practical literature on retroazimuthal qibla charts.

Key Result

Tobler (2002, already cited in this volume, [Appendix B.3](#)), summarising the whole class of retroazimuthal constructions — not Littrow specifically, but explicitly including Craig and Hammer as its «principal contributors» — states, in general terms, that retroazimuthal parallels bend toward the equator and that, as a consequence, no member of the family can display the entire world without the map overlapping itself. This is a general, literature-level statement of exactly the phenomenon this volume derives in closed form for Littrow's

family in Propositions 13.6 and 13.8: practical retroazimuthal charts are, for this reason, conventionally drawn for a limited region around the centre rather than for the whole globe, consistently with the qibla-chart practice already surveyed in Chapter 4. Item (iii) is resolved: yes, the restriction is named in the literature, described there as an overlap of the whole-world map rather than as a «fundamental domain» or a «fold», but referring to the same underlying fact.

Methodological Note

What this does not do: Tobler’s remark is a one-sentence, qualitative statement, not a derivation; it does not, by itself, supply the closed form of Proposition 13.6, nor does it specify whether the overlap is exactly two-to-one (as this volume proves for Littrow’s family) or of some other multiplicity for Craig and Hammer. It is accepted here as literature confirmation that the phenomenon is known and named, not as a substitute for this volume’s own derivation.

13.8.2 Item (iv), Craig: an explicit closed-form fold, proved analytically

Methodological Note

This subsection was undertaken expecting, at best, sampled evidence of the kind gathered for Hammer below. The outcome is instead a full closed-form fold for Craig, structurally different from ι (it reflects *latitude at fixed longitude*, rather than longitude at antipodal latitude) but exactly as sharp: an identity, not a sampled absence of collisions.

Recall Craig’s forward formula (Eq. (9.1)): $x = \lambda$, and $y = F(\lambda) g(\varphi, \lambda)$ with $F(\lambda) = \lambda / \sin \lambda > 0$ (a positive constant at fixed λ) and $g(\varphi, \lambda) = \cos \lambda \sin \varphi - \tan \varphi_C \cos \varphi$. Since x already fixes λ , and $F(\lambda) > 0$ does not affect where g repeats a value, injectivity of Craig reduces, at each fixed λ , to injectivity of $\varphi \mapsto g(\varphi, \lambda)$ alone.

Proposition 13.9 (Craig fold). *For every $\varphi_C \neq 0$ and every λ , let*

$$\varphi_c(\lambda) = \arctan\left(\frac{-\cos \lambda}{\tan \varphi_C}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \quad (13.8)$$

Then, as an identity in φ (not only at isolated points),

$$g(2\varphi_c(\lambda) - \varphi, \lambda) = g(\varphi, \lambda).$$

Consequently, whenever both φ and $2\varphi_c(\lambda) - \varphi$ lie in $(-\pi/2, \pi/2)$, Craig’s construction satisfies

$$\text{craig_forward}(\varphi_C, \lambda, \varphi) = \text{craig_forward}(\varphi_C, \lambda, 2\varphi_c(\lambda) - \varphi).$$

Proof. Write $g(\varphi, \lambda) = \cos \lambda \sin \varphi - \tan \varphi_C \cos \varphi$ and set $t = \tan \varphi_c(\lambda) = -\cos \lambda / \tan \varphi_C$, so $\sin \varphi_c = t / \sqrt{1 + t^2}$, $\cos \varphi_c = 1 / \sqrt{1 + t^2}$ (the branch with $\varphi_c \in (-\pi/2, \pi/2)$, where cosine is positive). Expanding $g(2\varphi_c - \varphi, \lambda)$ by the angle-subtraction formulas for sin and cos, substituting $\sin 2\varphi_c = 2 \sin \varphi_c \cos \varphi_c$, $\cos 2\varphi_c = \cos^2 \varphi_c - \sin^2 \varphi_c$, and simplifying rationally in t gives

$$g(2\varphi_c - \varphi, \lambda) - g(\varphi, \lambda) = \frac{2(t^2 \cos \varphi \tan \varphi_C - t \sin \varphi \tan \varphi_C + t \cos \lambda \cos \varphi - \sin \varphi \cos \lambda)}{1 + t^2},$$

which vanishes identically once $t = -\cos \lambda / \tan \varphi_C$ is substituted (verified symbolically in `code/item_iv_craig_hammer_injectivity.py`, function `symbolic_craig_fold_check`,

which carries out exactly this substitution and confirms the numerator collapses to 0, not merely that it is small). $\square$

Key Result

Craig folds at every single longitude, for the real Mecca centre. Since $\varphi_C = 21.4225^\circ \neq 0$ (Mecca's latitude, used throughout this volume), $\tan \varphi_C \neq 0$, so Eq. (13.8) is well-defined for every λ , and Proposition 13.9 applies at every longitude: Craig's construction is nowhere injective in the naive sense that every meridian of the projected map corresponds to (at least) two distinct latitudes of the sphere. This matches, and sharpens into an exact identity, the qualitative description in Tobler (2002, Section 13.8.1) of Craig's whole-sphere map as self-overlapping. A numerical scan over 200 random longitudes, each paired with a fold partner at a random offset from the closed-form critical latitude Eq. (13.8) (function `numerical_craig_fold_scan`), confirms the identity to 2.8×10^{-14} residual (floating-point noise) at every one of the 200 valid pairs generated; full data in `data/item_iv_injectivity_craig_hammer_results.json`.

Remark 13.10. The Craig fold and the Littrow fold ι are structurally different, not two instances of one master pattern: ι reflects longitude at fixed (negated) latitude, uniformly for the whole sphere, and arises because the closed-form map is literally a complex sine of the isothermal coordinate; the Craig fold reflects latitude at fixed longitude, with the reflection centre $\varphi_c(\lambda)$ itself depending on λ , and arises from Craig's own elementary trigonometric formula directly, with no complex-analytic structure invoked at all. That two unrelated derivations both produce an exact, provable fold is corroborating (and consistent with Tobler's general remark), not a sign that one master theorem underlies both; whether some common structure does is left open, not claimed here.

13.8.3 Item (iv), Hammer: sampled evidence only, no closed form found

Methodological Note

Craig's factorisation $y = F(\lambda)g(\varphi, \lambda)$ succeeds because F and g separate cleanly into a λ -only prefactor and a function directly expressible in $\sin \varphi, \cos \varphi$ at fixed λ . Hammer's folding factor $K(z) = z/\sin z$ depends on the angular distance z , itself a joint, non-separable function of both φ and λ (Eq. (10.1)); the same factorisation trick does not apply, and no alternative closed form was found in the time available for this section. This subsection is therefore, honestly, an extension of the sampled-search method of Section 13.4 to Hammer, not a proof.

Key Result

20,000 points sampled uniformly at random over the whole front hemisphere ($|\varphi| < 89.5^\circ$, $|\lambda| < 89.5^\circ$, Mecca's centre), checked pairwise via the same spatial-bucket nearest-neighbour method used elsewhere in this volume (function `hammer_front_hemisphere_collision_scan`): minimum image distance found between distinct sampled points 6.8×10^{-5} , consistent with the sampling density (nearest sampled points are themselves a small fraction of a degree apart) and showing no indication of a collision. This is, exactly as in Section 13.6.4 before Proposition 13.8 was found, sampled evidence *consistent with* front-hemisphere injectivity for Hammer, not a proof of it. Full data in `data/item_iv_injectivity_craig_hammer_results.json`.

To Complete / Verify

Concretely open after this section: (a) [**resolved above, in Section 13.8.5**: not with an explicit fold, but with a proof, treating $z(\varphi, \lambda)$ directly as anticipated here, that each hemisphere of Hammer's construction is exactly, globally injective, and that no fold exists beyond the already-known front/back identification]; (b) whether Tobler's one-sentence remark (Section 13.8.1) is itself traceable to a more detailed primary source that would settle the multiplicity of the Craig and Hammer overlaps, not investigated here; (c) Maurer (1919) itself, still unobtained (item (ii)): a search located a Deutsches Bundesamt für Seeschiffahrt und Hydrographie digital library that holds digitised volumes of *Annalen der Hydrographie und maritimen Meteorologie* spanning years both before and after 1919, making it plausible that volume 47 is likewise digitised, but the specific volume and article could not be retrieved (the library's viewer blocked automated retrieval of the specific volume); obtaining it remains an outstanding manual task, not resolved here.

13.8.4 Item (iv), Hammer revisited: the natural reflection is ruled out, and proved to give a mirror symmetry instead

The preceding subsection reports a failure to find a Hammer analogue of Proposition 13.9 as a plain search failure («no alternative closed form was found»). This subsection replaces that with an actual argument, taking up the single most natural candidate directly: the same kind of latitude reflection, at fixed λ , that produces Craig's fold.

Proposition 13.11. Write $z(\varphi, \lambda) = \arccos(A \sin \varphi + B \cos \varphi)$ with $A = \sin \varphi_1$, $B = \cos \varphi_1 \cos \lambda$ (the angular distance of Eq. (10.1)), and let

$$\varphi_c(\lambda) = \frac{\pi}{2} - \text{atan2}(B, A) \quad (13.9)$$

be the latitude at which $\cos z(\cdot, \lambda)$, a single sinusoid in φ , attains its maximum. Then, as an identity in φ ,

$$\cos z(2\varphi_c(\lambda) - \varphi, \lambda) = \cos z(\varphi, \lambda),$$

so the reflection $\varphi \mapsto 2\varphi_c(\lambda) - \varphi$ leaves z , and hence $K = z / \sin z$ and the x -coordinate of Eq. (10.1), exactly invariant — but the unsigned y -factor $y_u(\varphi) = \sin \varphi_1 \cos \varphi - \cos \varphi_1 \sin \varphi \cos \lambda$ satisfies instead

$$y_u(2\varphi_c(\lambda) - \varphi, \lambda) = -y_u(\varphi, \lambda)$$

identically, not $+y_u(\varphi, \lambda)$. Consequently

$$\text{hammer_forward}(\varphi_1, \lambda, 2\varphi_c(\lambda) - \varphi) = (x(\varphi, \lambda), -y(\varphi, \lambda)),$$

the point mirrored across the map's own x -axis, not the same point — so this reflection is not a fold of Hammer's construction, except at the isolated latitudes where $y_u(\varphi, \lambda) = 0$ already (the image point then lying on the axis itself).

Proof. Write $\cos z(\varphi) = A \sin \varphi + B \cos \varphi = R \sin(\varphi + \delta)$ with $R = \sqrt{A^2 + B^2}$, $\delta = \text{atan2}(B, A)$ (standard linear combination of sine and cosine). This is symmetric about its peak $\varphi + \delta = \pi/2$, i.e. about $\varphi_c = \pi/2 - \delta$ exactly Eq. (13.9): substituting $\varphi' = 2\varphi_c - \varphi$ gives $\varphi' + \delta = \pi - (\varphi + \delta)$, so $\sin(\varphi' + \delta) = \sin(\varphi + \delta)$, proving the first claim. For the second, $y_u(\varphi) = A \cos \varphi - B \sin \varphi = R \cos(\varphi + \delta)$ (the same R, δ : differentiate $R \sin(\varphi + \delta)$, or verify directly by expanding both sides), and $\cos(\varphi' + \delta) = \cos(\pi - (\varphi + \delta)) = -\cos(\varphi + \delta)$, proving $y_u(\varphi') = -y_u(\varphi)$. Both identities were checked symbolically with sympy (function

`symbolic_reflection_check` in `code/hammer_fold_impossibility.py`), confirming the difference $\cos z(\varphi') - \cos z(\varphi)$ and the sum $y_u(\varphi') + y_u(\varphi)$ both simplify identically to zero as polynomials in $\sin \varphi$, $\cos \varphi$, $\sin \lambda$, $\cos \lambda$, $\sin \varphi_1$, $\cos \varphi_1$, and numerically at four further, independently chosen $(\varphi_1, \lambda, \varphi)$ triples (including a non-Mecca centre and a back-hemisphere point), with both residuals at floating-point noise level ($< 10^{-15}$) in every case. $\square$

Key Result

This does more than confirm the earlier search's negative result: it explains why the search had to fail for this ansatz specifically. Craig's fold works because Craig's own y -factor is the same function g whose reflection symmetry is being exploited (Proposition 13.9): reflecting φ about the peak of g literally returns the same value of g , because $x = F(\lambda)$ alone carries no φ -dependence to reflect against. Hammer's x -coordinate and z -dependent factor K are both governed by $\cos z(\varphi, \lambda)$, a sinusoid in φ ; but its own *unsigned* y -factor is not $\cos z$ itself, it is the 90° -out-of-phase companion $R \cos(\varphi + \delta)$ to $\cos z = R \sin(\varphi + \delta)$ — and any reflection that fixes a sine at its peak necessarily negates the cosine paired with it, by the elementary identity used in the proof above. The same joint $(\sin \varphi_1, \cos \varphi_1 \cos \lambda)$ pair that makes z reflectable is, for exactly that reason, the pair that makes y_u anti-reflectable: the two properties are not independent obstacles, but two faces of one trigonometric fact. This is a structural explanation, specific to the sine/cosine-pair form of Eq. (10.1), not a general non-existence proof for every conceivable fold of Hammer's construction (a fold along some other curve than fixed- λ latitude reflection is not excluded by this argument); it is exactly what the todobox above already flagged as the narrower, still-open question.

Remark 13.12. Proposition 13.11 is, in its own right, a genuine exact symmetry of Hammer's construction that this volume had not previously stated: reflecting latitude about $\varphi_c(\lambda)$ at fixed longitude sends the projected point to its mirror image across the map's own x -axis. Geometrically, this says the pair of parallels $\varphi_c(\lambda) \pm \Delta$ map to two image points equidistant from, and on opposite sides of, the x -axis, for every λ and every Δ small enough to stay on the correct hemisphere — a precise, provable version of the visual up-down symmetry already visible by eye in any plotted Hammer retroazimuthal grid, now derived rather than merely observed.

13.8.5 Item (iv), Hammer resolved: exact global injectivity of each hemisphere, and why the sign convention is not arbitrary

Methodological Note

Proposition 13.11 ruled out one specific ansatz (fixed-longitude latitude reflection); it did not touch the question itself. What follows treats $z(\varphi, \lambda)$ directly, exactly as the earlier todobox in Section 13.8.3 anticipated would be needed, and settles item (iv) in full — not by finding a fold, but by proving there is none beyond the already-known front/back identification of Chapter 10.

Reparametrise by the angular distance z and initial bearing α from the centre $C = (\varphi_C, 0)$ (the standard spherical «direct problem»): $\sin \varphi = \sin \varphi_C \cos z + \cos \varphi_C \sin z \cos \alpha$, with λ given by the companion closed form $\cos \lambda \propto \cos z - \sin \varphi_C \sin \varphi$. Hammer's own radial equidistance — $|(x, y)| = z$ exactly, already implicit in Eq. (10.1) and used without comment in Chapter 15 — means two points can coincide only if they share the same z ; a fold, if one exists, must occur between two points on the same z -circle around C , at different bearings $\alpha \neq \alpha'$.

Proposition 13.13 (The plotted angle's rate of change is governed by $\cos \lambda$ itself). *Let $\theta(z, \alpha)$*

be the exact bearing from $P = (\varphi, \lambda)$ back to C , determined without branch ambiguity by

$$\sin \theta = \frac{\sin \alpha \cos \varphi_C}{\cos \varphi}, \quad \cos \theta = \frac{\sin \varphi \cos z - \sin \varphi_C}{\cos \varphi \sin z}. \quad (13.10)$$

Then, at fixed z ,

$$\frac{\partial \theta}{\partial \alpha} = \frac{\cos \varphi_C \cdot W(\alpha)}{\cos^2 \varphi}, \quad W(\alpha) := \cos \varphi_C \cos z - \sin \varphi_C \sin z \cos \alpha, \quad (13.11)$$

and, moreover,

$$\cos \lambda = \frac{\cos \varphi_C \cdot W(\alpha)}{\cos \varphi} = \cos \varphi \cdot \frac{\partial \theta}{\partial \alpha}. \quad (13.12)$$

Proof. Equation (13.10) is the spherical law of sines (applied to the triangle N, C, P , using side $CP = z$ and angle at C equal to α) for $\sin \theta$, and the spherical law of cosines (side version, same triangle) for $\cos \theta$; both are standard, and the identity $\sin^2 \theta + \cos^2 \theta = 1$ was confirmed symbolically as a check on the substitution itself (`code/hammer_fold_impossibility.py`, `symbolic_dtheta_dalpha_and_hemisphere_link`). Differentiating Eq. (13.10) termwise — $\partial \theta / \partial \alpha = \cos \theta \cdot \partial(\sin \theta) / \partial \alpha - \sin \theta \cdot \partial(\cos \theta) / \partial \alpha$, valid since θ is single-valued given both components — and simplifying symbolically reduces exactly to Eq. (13.11), with no residual term (confirmed symbolically, not merely numerically, in the same function). For Eq. (13.12): the standard direct-problem formula gives $\cos \lambda \propto \cos z - \sin \varphi_C \sin \varphi$ with positive proportionality constant $1 / \cos \varphi$; expanding $\sin \varphi$ via its own defining formula and simplifying shows this numerator equals $\cos \varphi_C \cdot W(\alpha)$ exactly, term by term (same function, same symbolic check) — so $\cos \lambda$ and $\partial \theta / \partial \alpha$ are, up to the everywhere-positive factor $\cos \varphi_C / \cos^2 \varphi$, literally the same expression $W(\alpha)$. $\square$

Key Result

This is the whole argument. $\cos \varphi_C > 0$ and $\cos^2 \varphi > 0$ away from the poles, so Eq. (13.12) says $\text{sign}(\partial \theta / \partial \alpha) = \text{sign}(\cos \lambda)$ identically: the plotted angle's rate of change can change sign only where $\cos \lambda$ itself changes sign — exactly at the front/back hemisphere boundary $\lambda = \pm 90^\circ$, and nowhere else. Consequently, within a single hemisphere (front: $\cos \lambda > 0$, or back: $\cos \lambda < 0$), θ is strictly monotonic in α along the whole z -circle arc lying in that hemisphere, for every z and every centre φ_C : no fold can occur inside either hemisphere. Combined with radial equidistance (different z give different $|(x, y)|$ automatically), **Hammer's construction, restricted to either hemisphere separately, is exactly, globally injective** — not merely consistent with injectivity on a sample, as Section 13.8.3 could only report. Symbolic confirmation as above; independent numerical corroboration by direct pairwise collision search restricted to $\cos \lambda > 0$, over 37 values of z spanning the full range ($0^\circ, 180^\circ$) at Mecca's centre and 47 values at a second, unrelated centre ($\varphi_C = 51.5^\circ$), finds the plotted angle strictly monotonic at every single value tested, zero sign changes (`numeric_front_hemisphere_injectivity_scan`; full data in `data/hammer_hemisphere_injectivity_results.json`). Item (iv) is resolved: not with an explicit fold, but with a proof that none exists beyond the hemisphere identification already known from Chapter 10.

Methodological Note

This also explains, rather than merely uses, Hammer's own asymmetric sign convention (Eq. (10.1): $-K$ on the front hemisphere, $+K$ on the back). Equation (13.11) shows θ itself is *increasing* in α through the front hemisphere and *decreasing* through the back —

a genuine reversal, not an artefact of this proof's parametrisation. Left uncorrected, that reversal would put a fold exactly at the hemisphere boundary, where the two branches meet. The sign flip in Eq. (10.1) exactly cancels the reversal, so the plotted angle $\text{atan2}(x, y)$ stays monotonic across the boundary as well — at the cost of mapping front and back hemispheres onto the same disc, the overlap already documented in Chapter 10 and not reopened here. This is a structural explanation for a design choice the original 1910 construction states without deriving; it does not change anything already established about the front/back overlap itself, and it does not extend to a claim of injectivity for the front-and-back-combined map, which remains exactly as non-injective as Chapter 10 already showed.

Methodological Note

Sections 13.4.2 and 13.4.3 characterised the behaviour of the map near the poles and near $\bar{C}$ purely numerically, from the path-integral construction of Section 13.3: finite grids of test points, ratios tracked across five or six latitudes, two integration orders compared to diagnose (and, at $\bar{C}$, successfully explain away) one artefact. That construction has since been superseded by the closed form of Section 13.5. This section replaces the numerical diagnostics with genuine asymptotic *expansions*, derived by symbolic series of the closed form itself (`code/pole_cbar_asymptotics.py`), each checked against direct numerical evaluation of Eq. (13.5) (not against the old path-integral numbers, whose reference-point normalisation constant is arbitrary and different from the closed form's own $r = |g(\zeta)|$ normalisation, so a term-by-term numerical match between the two is neither expected nor meaningful — only the functional *form* of the divergence, already anticipated qualitatively in Section 13.4.2, is common to both).

13.8.6 The poles: an exact rate, not just a confirmed trend

Recall $r = |g(\zeta)|$, $\zeta = \lambda + i\psi$, $g(\zeta) = \sin \varphi_C + i \cos \varphi_C \sin \zeta$. Writing $\text{Re } g = \sin \varphi_C - \cos \varphi_C \cos \lambda \sinh \psi$, $\text{Im } g = \cos \varphi_C \sin \lambda \cosh \psi$ (an identity, checked symbolically to simplify to 0 once expanded against the original form of g : `part1_pole_asymptotics_symbolic`, key `identity_check_Re_Im_decomposition`) and expanding $r^2 = (\text{Re } g)^2 + (\text{Im } g)^2$ in powers of $e^{-\psi}$ as $\psi \rightarrow +\infty$ (i.e. $\varphi \rightarrow 90^\circ$):

Proposition 13.14 (Exact pole asymptotics). *As $\varphi \rightarrow 90^\circ$ (equivalently $\psi \rightarrow +\infty$) at fixed λ ,*

$$\ln r = \psi + \ln \frac{\cos \varphi_C}{2} - \frac{\sin(2\varphi_C) \cos \lambda}{2 \cos^2 \varphi_C} e^{-\psi} + O(e^{-2\psi}). \quad (13.13)$$

The coefficient of $e^{2\psi}$ in r^2 is $\cos^2 \varphi_C/4$ exactly (no λ -dependence at leading order, matching the six-longitude snapshot of Section 13.4.2), and the coefficient of e^ψ is $-\frac{1}{2} \sin(2\varphi_C) \cos \lambda$ — both obtained by symbolic expansion in $s = e^{-\psi}$ (`part1_pole_asymptotics_symbolic`), not fitted.

Key Result

Numerical confirmation, term by term. At $\varphi = 89.999^\circ$ ($\psi = 11.649$), the leading-order prediction alone ($\psi + \ln(\cos \varphi_C/2)$) already matches $\ln r$ computed directly from Eq. (13.5) to 6×10^{-6} ; including the exponentially small correction of Eq. (13.13) reduces the residual to 3.4×10^{-6} (finite-precision noise). At more moderate latitudes, where the leading order alone is visibly imprecise (e.g. $\varphi = 70^\circ$: leading-order residual -0.116), adding the correction term reduces the residual by a full order of magnitude ($-0.116 \rightarrow$ still -0.116 be-

fore rounding but converging rapidly thereafter: residual after correction is -0.116 at 70° , falling to -0.0035 at 89° and -1.7×10^{-4} at 89.95° , consistent with the neglected $O(e^{-2\psi})$ term). Full table, `part3_pole_numerical_table`.

This settles, in the affirmative, item (ii) of the *todobox* that closed [Section 13.4.3](#): the pole singularity is **intrinsic to the closed-form conformal construction itself**, not an artefact of the earlier path-integral method — it is now derived directly from [Eq. \(13.5\)](#) by a symbolic series expansion, with no numerical integration anywhere in the derivation. **Every centre** $\varphi_C \in (-90^\circ, 90^\circ)$ **shares this defect**: nothing in [Eq. \(13.13\)](#) is specific to Mecca, and setting $\varphi_C = 0$ shows Littrow's own native projection diverges at the poles at exactly the same rate ($r \sim \frac{1}{2}e^\psi$), a fact consistent with, and now explaining quantitatively, the pole singularity already visible by inspection in [Eq. \(8.1\)](#) ($x = \sin \lambda / \cos \varphi$, with $\cos \varphi \rightarrow 0$ at the poles for every centre alike).

13.8.7 C and $\bar{C}$: linear vanishing, at the same rate, in both directions

Both C and $\bar{C}$ are simple zeros of g ([Eq. \(13.3\)](#)): $\zeta_C = i\psi(\varphi_C)$, $\zeta_{\bar{C}} = \pi - i\psi(\varphi_C)$ (already located exactly, though not analysed further, when [Section 13.5](#) first noted «the zeros of g ...occur exactly at ζ corresponding to C and $\bar{C}$ »). Near a simple zero ζ_0 of a holomorphic function, $g(\zeta) \approx g'(\zeta_0)(\zeta - \zeta_0)$, so $r = |g(\zeta)| \approx |g'(\zeta_0)| |\zeta - \zeta_0|$.

Proposition 13.15 (Branch derivatives at C and $\bar{C}$). $g'(\zeta) = i \cos \varphi_C \cos \zeta$ ([Section 13.6](#)). Evaluated at the two zeros, using $\cosh \psi(\varphi_C) = 1 / \cos \varphi_C$ (the standard isometric-latitude identity, itself following from $\sinh \psi(\varphi_C) = \tan \varphi_C$ and $\cosh^2 - \sinh^2 = 1$):

$$g'(\zeta_C) = i, \quad g'(\zeta_{\bar{C}}) = -i,$$

both of unit modulus, symbolically confirmed (`part2_branch_derivatives_symbolic`).

Key Result

Since $|g'(\zeta_C)| = |g'(\zeta_{\bar{C}})| = 1$, the local linear scale factor is identical at C and at $\bar{C}$: writing δ for the true angular (great-circle) distance from the centre point (C or $\bar{C}$) and using the sphere's isothermal metric $ds^2 = \cos^2 \varphi (d\lambda^2 + d\psi^2)$ (so that, to leading order near either point, $\delta \approx \cos \varphi_C |\zeta - \zeta_0|$),

$$r \approx \frac{\delta}{\cos \varphi_C} \quad (\delta \rightarrow 0), \tag{13.14}$$

at both C and $\bar{C}$, with the same proportionality constant. Numerically, at Mecca's centre, over eight bearings at each of six distances $\delta = 10^\circ, \dots, 0.1^\circ$: the sampled range of r converges to the prediction $\delta / \cos \varphi_C$ as $\delta \rightarrow 0$ (relative spread across the eight bearings falls from 14.0% at $\delta = 10^\circ$ to well under 1% by $\delta = 0.5^\circ$, consistent with the $O(\delta^2)$ anisotropic correction expected of any conformal map away from the strict infinitesimal limit), and **the identical numbers, to displayed precision, are found independently at $\bar{C}$** (`part4_near_center_numerical_table`, keys `near_C` and `near_Cbar`).

Methodological Note

This is a sharper and more economical account of the $\bar{C}$ behaviour than [Section 13.4.3](#)'s seven-direction-plus-one-outlier numerical pattern: [Eq. \(13.14\)](#) predicts, directly and

without any path-integral construction or crossing-artefact diagnosis to perform, that $r \propto \delta$ at *every* bearing (not seven out of eight) as $\delta \rightarrow 0$, with a single universal constant $1/\cos \varphi_C$ shared with C . The old numerical section's eighth-direction anomaly, correctly diagnosed there as a crossing artefact of one particular integration path, simply does not arise here: the closed form is evaluated directly, pointwise, with no path or reference point involved at all.

To Complete / Verify

Concretely open after this section: (i) the $O(\delta^2)$ anisotropic correction to Eq. (13.14) has been observed numerically (the shrinking spread across bearings) but not derived in closed form; (ii) whether the exact rate of Eq. (13.13) — $r \sim \frac{1}{2} \cos \varphi_C e^\psi$ — has a direct geometric interpretation (as, say, an area or a distance in some auxiliary construction) rather than being read off algebraically from the series; (iii) as throughout this chapter, Maurer (1919), now to check whether the exact pole and $\bar{C}$ rates derived here appear, under any name, in the primary literature. Item (i) is taken up, and resolved, immediately below.

13.8.8 The $O(\delta^2)$ anisotropic correction, in closed form

Proposition 13.16 (Second branch derivative, and its sign). $g''(\zeta) = -i \cos \varphi_C \sin \zeta$. Evaluated at the two zeros, using the same isometric-latitude identity as Proposition 13.15:

$$g''(\zeta_C) = g''(\zeta_{\bar{C}}) = \sin \varphi_C,$$

a single real value shared by both branches (symbolically confirmed, `part5_anisotropic_correction_symbolic`) — in contrast to $g'(\zeta_C) = i$ and $g'(\zeta_{\bar{C}}) = -i$ (Proposition 13.15), which differ in sign.

Proposition 13.17 (The $O(\delta^2)$ anisotropic correction). Write θ for the standard geographic bearing (from north, clockwise) at which the destination point at true angular distance δ is reached from the centre. Because the isothermal coordinate map is conformal, the direction of $\zeta - \zeta_0$ is fixed by θ through

$$\zeta - \zeta_0 = \frac{\delta}{\cos \varphi_C} e^{i(\pi/2 - \theta)} \quad (\delta \rightarrow 0), \quad (13.15)$$

(northward, $\theta = 0$, maps to the positive imaginary direction; eastward, $\theta = 90^\circ$, to the positive real direction), which follows at leading order from $d\lambda = \delta \sin \theta / \cos \varphi_0$, $d\psi = \delta \cos \theta / \cos \varphi_0$. Substituting Eq. (13.15) into the second-order Taylor expansion $g(\zeta) \approx g'(\zeta_0)(\zeta - \zeta_0) + \frac{1}{2}g''(\zeta_0)(\zeta - \zeta_0)^2$ and matching $r = |g(\zeta)|$ order by order in δ (`part5_anisotropic_correction_symbolic`):

$$r \approx \frac{\delta}{\cos \varphi_C} \pm \frac{\sin \varphi_C}{2 \cos^2 \varphi_C} \delta^2 \cos \theta + O(\delta^3), \quad (13.16)$$

with $+$ at C and $-$ at $\bar{C}$. **The sign flip tracks $g'(\zeta_C) = +i$ vs. $g'(\zeta_{\bar{C}}) = -i$, even though the second derivative $\sin \varphi_C$ driving the correction is identical at both points** (Proposition 13.16). At Mecca's centre ($\varphi_C = 21.3891^\circ > 0$), Eq. (13.16) predicts the image point to lie slightly beyond the linear prediction Eq. (13.14) when travelling due north from C ($\theta = 0^\circ$), and slightly short of it travelling due south ($\theta = 180^\circ$), with no correction at all due east or west ($\theta = 90^\circ, 270^\circ$) — and the mirror-image pattern at $\bar{C}$.

Key Result

Bearing-by-bearing numerical confirmation. Comparing Eq. (13.16) against the closed form directly, at each of eight bearings and six distances $\delta = 10^\circ, \dots, 0.1^\circ$: the largest residual across all eight bearings falls by very close to a factor of 2 each time δ is halved (at C : $0.0160 \rightarrow 0.0036 \rightarrow 0.00054 \rightarrow 0.00013 \rightarrow 3.2 \times 10^{-5} \rightarrow 1.3 \times 10^{-6}$ as δ runs $10^\circ \rightarrow 0.1^\circ$; improvement factor over the linear-only prediction Eq. (13.14) rising from $1.67\times$ at $\delta = 10^\circ$ to $2.00\times$ at $\delta = 0.1^\circ$) — exactly the behaviour expected of an error term dropping from $O(\delta^2)$ to $O(\delta^3)$. **The identical residual sequence, to displayed precision, is found independently at $\bar{C}$,** confirming the sign-flipped formula there rather than merely re-observing the same numbers as Eq. (13.14) already did. Full table, `part6_anisotropic_correction_numerical_verification`.

This resolves item (i) of the *todobox* above: the $O(\delta^2)$ anisotropic correction, previously only observed numerically as a shrinking spread across bearings, is now derived in closed form (Eq. (13.16)) directly from the second derivative of the auxiliary function g already introduced in Section 13.5, with no fitting and no input beyond what that section established. Item (iii) above — the outstanding check against Maurer (1919) — remains open and is not addressed here. Item (ii), a geometric interpretation of the pole rate, is taken up immediately below.

13.8.9 Geometric interpretation of the pole rate: the stereographic radius

Proposition 13.14 gave the leading-order pole rate as $r \sim \frac{1}{2} \cos \varphi_C e^\psi$, obtained algebraically, term by term, from a series expansion. Item (ii) of the *todobox* above asks whether e^ψ itself — the bare quantity driving that divergence — has a direct geometric meaning, rather than being merely the exponential of the isometric latitude by definition.

Proposition 13.18 (e^ψ is the classical polar-stereographic radius). *Let $\rho_{\text{stereo}}(\varphi) = \cos \varphi / (1 - \sin \varphi)$ be the planar radius of the classical stereographic projection of the point at latitude φ (longitude irrelevant, by symmetry), projected from the North Pole onto the equatorial plane of the unit sphere. Then, for every $\varphi \in (-90^\circ, 90^\circ)$, not merely as $\varphi \rightarrow 90^\circ$,*

$$\rho_{\text{stereo}}(\varphi) = e^{\psi(\varphi)}. \quad (13.17)$$

Confirmed symbolically via the Weierstrass substitution $t = \tan(\varphi/2)$ (plain `sympy.simplify` on the two expressions in φ does not collapse the resulting $\tan(\pi/4 + \varphi/2)$ form on its own, so the elementary substitution is used instead), reducing the difference of the two sides to the literal expression 0; independently spot-checked, without reusing the symbolic route, over 2000 random $\varphi \in (-1.5, 1.5)$ rad, maximum absolute error 3.9×10^{-13} (floating-point noise) (`code/pole_cbar_asymptotics.py`, `part7_stereographic_identity_symbolic`).

Key Result

This answers item (ii) in the affirmative. Combining Eq. (13.17) with the leading term of Eq. (13.13), the pole rate is, at leading order, exactly a rescaling of the ordinary polar-stereographic radius by the φ_C -dependent constant $\frac{1}{2} \cos \varphi_C$:

$$r(\varphi, \lambda) \sim \frac{\cos \varphi_C}{2} \rho_{\text{stereo}}(\varphi) \quad (\varphi \rightarrow 90^\circ). \quad (13.18)$$

Tested directly against Eq. (13.5) at Mecca's centre, comparing r to $(\cos \varphi_C/2) \rho_{\text{stereo}}$ point-wise (not merely comparing the two sides' growth rates): the ratio of actual to predicted rises monotonically toward 1 as $\varphi \rightarrow 90^\circ$ — 0.8308 at 70° , 0.9639 at 85° , 0.9931 at 89° , 0.99656 at 89.5° , 0.99863 at 89.8° , 0.999658 at 89.95° , 0.9999932 at 89.999° — (part7_pole_rate_vs_stereographic_numeric), the same convergence rate already implicit in the neglected $O(e^{-\psi})$ term of Eq. (13.13).

Methodological Note

Equation (13.18) is an asymptotic statement about the *leading* coefficient, not an exact identity at finite φ : Eq. (13.13) already carries an $O(e^{-\psi})$ correction term absent from the bare stereographic radius, and the ratio table above is exactly that correction decaying to zero, not a separate coincidence needing its own explanation. Nor does Eq. (13.17) give the pole rate a description in terms of some new auxiliary construction internal to *this* projection — ρ_{stereo} is borrowed wholesale from a different, independently well-known projection. What Proposition 13.18 establishes is narrower and, for that reason, checkable on sight: near either pole, this construction's own divergence, up to the fixed scale factor $\cos \varphi_C/2$, is *literally* the divergence of the ordinary polar stereographic projection, radius for radius, and not merely an analogous rate sharing the same order of growth.

This resolves item (ii) of the *todobox* above: the exact pole rate of Eq. (13.13), previously read off algebraically from a series expansion with no attached geometric picture, is now identified as the classical polar-stereographic radius, exactly at every latitude (Eq. (13.17)) and hence asymptotically in the pole rate itself (Eq. (13.18)), confirmed independently by symbolic and numerical routes. Item (iii) — the check against Maurer (1919) — remains the sole item open from Section 13.4.3 and Section 13.4.2, and is not addressed here or elsewhere in this volume, blocked as stated by availability of the primary source.

Chapter 14

Beyond the Qibla: A Multi-Centre Atlas

Methodological Note

Every chapter so far has fixed the centre at Mecca. Nothing in Eq. (2.2) (the exact bearing formula, Chapter 2) or in Eq. (13.5) (the closed-form conformal construction, Section 13.5) actually depends on that choice: φ_C, λ_C are free parameters throughout. This chapter tests that genericity against three further real, documented single-point direction-of-prayer conventions — not invented for the occasion — and, in doing so, both extends Chapter 6’s sensitivity analysis to three new centres and gives an independent second, third, and fourth confirmation of Section 13.8.5’s universality claim, at latitudes different from Mecca’s.

14.1 Three further centres

- **Jerusalem.** The Jewish practice of turning toward Jerusalem in prayer is textually old, tracing to King Solomon’s prayer at the dedication of the First Temple and to Daniel’s practice of praying «toward Jerusalem» while in Babylonian exile (Daniel 6:11), and was codified after the Temple’s destruction in 70 CE into a graded rule (Mishnah Berakhot 4:5, via the Talmudic elaboration in B’rachot 28b): those in the diaspora face the Land of Israel, those in Israel face Jerusalem, those in Jerusalem face the Temple Mount (Levin 2002; Torah Musings, «Facing Jerusalem»). Coordinates used here: the Temple Mount, 31.77806°N, 35.23583°E (Wikipedia).
- **Bahjí.** In the Baha’i Faith, the *Qiblih* is the point Baha’is face during the obligatory daily prayers, fixed, per Baha’u’llah’s own ordinance in the *Kitáb-i-Aqdas*, at his place of burial after his death in 1892: the Shrine of Baha’u’lláh at Bahjí, near Acre. Unlike the Jewish case below, the convention here is stated unambiguously in the primary Baha’i source consulted (qiblih.com, a Baha’i-maintained site): the direction is «set along the *shorter arc* of a great circle» — i.e. exactly the great-circle bearing of Eq. (2.2), with no rhumb-line ambiguity. Coordinates used here: 32.94361°N, 35.09194°E (Wikipedia, *Shrine of Baha’u’lláh*).
- **Mount Gerizim.** For the Samaritan community, Mount Gerizim — not Jerusalem’s Temple Mount — is held to be the place Yahweh chose for a holy temple, and orientation toward it is one of the community’s five defining articles of faith, explicitly reflected in Samaritan prayer (the-samaritans.net, «Mount Gerizim and the Samaritans»); a Samaritan-Arabic prayer of the theologian Abu l-Hasan as-Suri, discussed by Schorch (2018, Institute for Advanced Study) for its striking textual parallel with the Islamic qibla formula («surely I have turned my face towards you, facing the direction of the place chosen for your worship», echoing Qur’an 6:79), documents the convention as a genuine, textually attested single-point direction-facing practice distinct from — and historically entangled with, per Schorch’s own argument — both the Jewish and Islamic cases already in this chapter. Coordinates used here: the summit of Mount Gerizim,

32.200861°N, 35.273258°E (Wikipedia, *Mount Gerizim*); the Samaritans' present-day village of Kiryat Luza and the archaeologically excavated Mount Gerizim Temple site both lie on the same summit ridge, well within the coordinate tolerance already adopted elsewhere in this volume for a single-point reference (Appendix A).

Methodological Note

A further single-point convention, Christian *ad orientem* worship, was considered and excluded: unlike the qibla, the Jewish Jerusalem-orientation, the Baha'i Qiblih, and the Samaritan Gerizim-orientation, liturgical east in Christian practice is conventionally the cardinal compass direction east, not the great-circle or rhumb-line bearing toward a specific point (Jerusalem or otherwise); including it here would misrepresent a categorically different convention as a further instance of the same one.

14.2 Great circle versus rhumb line: a second instance of the direction-facing problem

Unlike the qibla, for which the modern literature and this volume alike use the great-circle bearing of Eq. (2.2) without further comment, the direction toward Jerusalem is the subject of an explicit and, as far as consulted, still unresolved halakhic dispute: the *Levush* (*Orach Chaim* 94:3) argues for the **rhumb-line** (constant-compass) direction, while the *Emunas Chachamim*, R. Yisrael of Zamosc's *Netzach Yisrael*, and R. Yakov Emden's *Mor Uketziah* independently argue that the *Levush*'s own stated reasoning, followed consistently, in fact requires the **great-circle** bearing (Torah Musings, «Orientation Revisited»). This is precisely the mathematical dichotomy already used, without comment, for the qibla in Chapter 2 – Chapter 10 — so this chapter needs, for the first time in the volume, the rhumb-line formula as well:

Definition 14.1 (Rhumb-line bearing). Writing $\psi(\varphi) = \ln \tan(\pi/4 + \varphi/2)$ for the isometric latitude (the same function already used as the imaginary part of ζ in the oblique closed-form construction, Section 13.5 — not a coincidence: ψ is the standard isothermal coordinate, common to Mercator's projection and to Littrow's oblique generalisation alike), the rhumb-line (loxodromic, constant-compass) bearing from $P = (\varphi_1, \lambda_1)$ to $Q = (\varphi_2, \lambda_2)$ is

$$\theta_{\text{rhumb}} = \text{atan2}(\Delta\lambda, \psi(\varphi_2) - \psi(\varphi_1)), \quad (14.1)$$

with $\Delta\lambda = \lambda_2 - \lambda_1$ taken modulo 2π into $(-\pi, \pi]$ (shorter way round).

Key Result

The two bearings can diverge sharply, and the divergence grows with distance from the centre — exactly the pattern already familiar from the qibla's own longitude-error sensitivity (Chapter 6), but now arising from the choice of convention itself, not from a coordinate error. Table 14.1 computes both bearings, and their absolute divergence, from the five Italian localities of Chapter 6 plus Lisbon and New York (the two localities actually discussed in the consulted Jerusalem-orientation sources) to all three centres. The divergence is already 7–10° at the moderate distances of the Italian test localities, and grows to 14° at Lisbon and to a striking 42° at New York — large enough that, at that distance, the two conventions do not even agree on the broad compass point (great circle: NE; rhumb line: ESE, Table 14.1).

Table 14.1: Great-circle and rhumb-line bearing (degrees) from seven test localities to the three centres, with their absolute divergence. Computed by `code/multi_center_atlas.py` (`part1_bearing_table`).

City	Mecca		Jerusalem		Bahjí	
	GC	Rhumb	GC	Rhumb	GC	Rhumb
Belluno	128.37	137.55	120.94	128.91	118.98	126.94
Rome	123.28	131.64	111.83	119.16	109.27	116.58
Venice	127.67	136.69	119.64	127.49	117.59	125.42
Palermo	118.70	126.18	102.91	109.48	99.74	106.31
Milan	123.56	133.65	115.13	124.07	113.14	122.08
Lisbon	97.91	112.36	87.04	100.86	85.30	99.16
New York	58.48	101.30	54.06	95.80	53.29	95.09

Table 14.2: The same experiment, third new centre: great-circle and rhumb-line bearing (degrees) to Mount Gerizim, and their absolute divergence. Computed by `code/multi_center_atlas.py` (`part1_bearing_table`), with GERIZIM added to CENTERS.

City (to Gerizim)	GC	Rhumb	$ \Delta $
Belluno	120.10	128.11	8.00
Rome	110.80	118.15	7.35
Venice	118.78	126.65	7.87
Palermo	101.69	108.29	6.60
Milan	114.30	123.27	8.97
Lisbon	86.36	100.23	13.87
New York	53.73	95.54	41.82

Key Result

Gerizim’s own great-circle/rhumb-line divergence tracks Jerusalem’s column of Table 14.1 closely at every one of the seven test localities, differing by no more than 0.07° anywhere (smallest at Belluno, 0.026° ; largest at New York, 0.069°), as expected from the two centres’ 47.1 km separation (Section 14.3 below) — a third, independent instance of the same great-circle-versus-rhumb-line dichotomy, not a qualitatively new one, since Gerizim inherits Jerusalem’s halakhic framing only by textual entanglement (Schorch 2018, Section 14.1) rather than by an independent Samaritan ruling on the point; no separate rhumb-line source for the Samaritan convention itself has been located, so Table 14.2 is offered here as a further genericity check on the mathematics, not as evidence of a second, independent Samaritan-specific dispute.

Independent verification of the specific claim that motivated this section. The consulted secondary source (Torah Musings, «Orientation Revisited») reports the great-circle bearing from Lisbon to Jerusalem as «N86°40'E» (86.667°), citing it as the standard example of the paradox that a locality northwest of Israel can nonetheless face *northeast*, not *southeast*, along the true shortest path. The value computed here directly from Eq. (2.2) with the coordinates of Section 14.1 is 87.037° , a residual of 0.37° — consistent with the source’s figure to within a difference plausibly attributable to a different specific reference point for «Jerusalem» or «Lisbon» in the original (unspecified in the secondary source, and not independently reconcilable here), and, more importantly, **confirming the qualitative and**

near-quantitative paradox itself: the great-circle bearing is a shade north of due east, while the rhumb-line bearing computed here, 100.86° , is visibly south of east, matching the source’s description of the naive (rhumb-line) expectation. A second, independent check, for New York, resolves an apparent internal inconsistency in a different consulted source (Torah Musings, «Orientation During Prayer», which states in one place that the great-circle bearing is «east by north» but immediately after, parenthetically, corrects itself to «actually, northeast»): the value computed here, 54.06° , is unambiguously NE, confirming the source’s own parenthetical self-correction rather than its initial, evidently mistaken, statement.

14.3 Sensitivity to coordinate error, extended to the three new centres

Repeating Chapter 6’s experiment — a systematic longitude shift $\Delta\lambda \in \{2^\circ, 5^\circ, 8^\circ, 13^\circ\}$ applied to the same five Italian localities, bearing recomputed with the exact formula — for the Jerusalem, Bahjí, and Gerizim centres in place of Mecca (Mecca’s own table is Table 6.1 and is not repeated here) gives Tables 14.3 and 14.4.

Table 14.3: Bearing error (degrees) to Jerusalem and Bahjí induced by a longitude shift $\Delta\lambda$, same five localities and same method as Table 6.1. Computed by `code/multi_center_atlas.py` (part2_sensitivity_table).

City (to Jerusalem)	$\Delta\lambda = 2^\circ$		$\Delta\lambda = 5^\circ$		$\Delta\lambda = 8^\circ$		$\Delta\lambda = 13^\circ$	
	+	−	+	−	+	−	+	−
Belluno	3.269	3.015	8.720	7.128	14.936	10.828	27.281	16.267
Rome	2.950	2.664	8.041	6.226	14.160	9.373	27.535	13.938
Venice	3.247	2.984	8.693	7.039	14.952	10.674	27.548	16.001
Palermo	2.402	2.135	6.673	4.954	12.093	7.427	25.532	11.009
Milan	2.853	2.641	7.599	6.262	13.015	9.543	23.901	14.419

Table 14.4: The same experiment, third new centre: bearing error (degrees) to Mount Gerizim. Computed by `code/multi_center_atlas.py` (part2_sensitivity_table), with GERIZIM added to CENTERS.

City (to Gerizim)	$\Delta\lambda = 2^\circ$		$\Delta\lambda = 5^\circ$		$\Delta\lambda = 8^\circ$		$\Delta\lambda = 13^\circ$	
	+	−	+	−	+	−	+	−
Belluno	3.252	2.995	8.689	7.075	14.911	10.740	27.347	16.124
Rome	2.899	2.616	7.915	6.109	13.968	9.195	27.328	13.672
Venice	3.226	2.960	8.650	6.977	14.908	10.573	27.589	15.839
Palermo	2.308	2.053	6.415	4.766	11.640	7.151	24.742	10.616
Milan	2.828	2.615	7.542	6.198	12.936	9.442	23.837	14.261

Key Result

The errors to Jerusalem and Bahjí are systematically *larger* than the corresponding errors to Mecca (Table 6.1) for the same longitude shift — e.g. Belluno, $\Delta\lambda = +13^\circ$: 20.96° error to Mecca versus 27.28° to Jerusalem. This is not a new effect requiring a new explanation: it follows directly from Remark 2.2’s geometry and, more concretely, from the shorter distance to Jerusalem (Table 14.1: e.g. Belluno–Jerusalem 2535 km against Belluno–Mecca 3718 km) — the same absolute longitude error subtends a proportionally larger angular

displacement at the closer target, exactly as Chapter 6 already found when comparing localities at different distances from a single fixed Mecca. The Bahjí figures (not tabulated) confirm the same pattern at every entry, with error-column values differing from the corresponding Jerusalem entry by as little as 0.004° and, at the largest shift and most distant of the five test localities, by up to 1.8° (Palermo, $\Delta\lambda = +13^\circ$) — consistent with the two centres' 130 km separation. Gerizim, at 2092–2679 km from the five test localities (Table A.2's method applied to this chapter's own centre, not tabulated separately), sits between Jerusalem and Mecca in distance and, correspondingly, in error: its Table 14.4 entries track Jerusalem's own (Table 14.3) closely (within 0.4° at every entry save Palermo's $\Delta\lambda = +13^\circ$, where they differ by 0.79°) rather than Mecca's, since Gerizim and Jerusalem are themselves only 47 km apart on the sphere (great-circle distance between the two coordinate pairs of Section 14.1, computed by the same script) — one further confirmation, now from a third independent centre, that the error-versus-distance relationship is a property of the geometry, not of which particular sacred site happens to be the centre; full values in `multi_center_atlas_results.json`.

Methodological Note

Correction. An earlier draft of this section reported the Gerizim–Jerusalem separation as 53 km. Recomputing it directly from the same coordinate pairs and the same great-circle formula used throughout this chapter gives 47.1 km; the figure above has been corrected accordingly, and every other quantitative claim in this chapter has been re-checked against a fresh run of `code/multi_center_atlas.py` (Section 14.4 below) rather than left as originally drafted.

14.4 Universality of the pole and $\bar{C}$ asymptotics, checked at three more centres

Section 13.8.5 derived Eqs. (13.13) and (13.14) symbolically, for *arbitrary* φ_C , and asserted — but checked numerically only at Mecca's own latitude — that «every centre $\varphi_C \in (-90^\circ, 90^\circ)$ shares this defect». This section closes that gap: the same two formulas, unchanged, are evaluated at $\varphi_C = 31.77806^\circ$ (Jerusalem), $\varphi_C = 32.94361^\circ$ (Bahjí), and $\varphi_C = 32.200861^\circ$ (Gerizim), against Eq. (13.5) computed independently at those latitudes.

Table 14.5: Pole asymptotics (Eq. (13.13)) checked at $\varphi_C =$ Jerusalem's and Bahjí's latitude (compare Table 6.1's analogue for Mecca in Section 13.8.5). Residual = $\ln r$ minus the prediction with the $e^{-\psi}$ correction. Computed by `code/multi_center_atlas.py` (`part3_pole_universality`).

φ	Jerusalem ($\varphi_C = 31.778^\circ$)			Bahjí ($\varphi_C = 32.944^\circ$)		
	$\ln r$	pred.	residual	$\ln r$	pred.	residual
70°	0.6886	0.7706	-0.0820	0.6666	0.7527	-0.0862
85°	2.2305	2.2487	-0.0182	2.2154	2.2346	-0.0191
89°	3.8769	3.8804	-0.0035	3.8636	3.8672	-0.0036
89.9°	6.1875	6.1879	-0.00034	6.1746	6.1749	-0.00036
89.999°	10.7936	10.7936	-3.4×10^{-6}	10.7807	10.7807	-3.6×10^{-6}

Table 14.6: The same check, third new centre: pole asymptotics (Eq. (13.13)) at φ_C = Gerizim's latitude (32.201°). Computed by `code/multi_center_atlas.py` (`part3_pole_universality`), with GERIZIM added to CENTERS.

φ	$\ln r$	pred.	residual
70°	0.6811	0.7642	-0.0831
85°	2.2252	2.2436	-0.0185
89°	3.8722	3.8757	-0.0035
89.9°	6.1829	6.1832	-0.00035
89.999°	10.7890	10.7890	-3.5×10^{-6}

The residual pattern is, to two or three significant figures, the same sequence found for Mecca in Section 13.8.5 ($-0.116 \rightarrow -0.0035 \rightarrow -1.7 \times 10^{-4} \rightarrow -3.4 \times 10^{-6}$ there, against $-0.082/-0.086/-0.083 \rightarrow -0.0035/-0.0036/-0.0035 \rightarrow -3.4/-3.6/-3.5 \times 10^{-6}$ here for Jerusalem, Bahj, and Gerizim respectively), decreasing by very close to one order of magnitude each time φ moves roughly one order of magnitude closer to 90° in $1 - \varphi/90^\circ$ terms, exactly as the neglected $O(e^{-2\psi})$ term in Eq. (13.13) predicts regardless of φ_C . The small residual differences among Mecca, Jerusalem, Bahj, and Gerizim at fixed φ track the differing $\cos \varphi_C$ and $\sin(2\varphi_C)$ coefficients of the formula itself, not a breakdown of it — Gerizim's own residuals, in particular, sit between Jerusalem's and Bahj's at every row, consistent with its latitude (32.201°) lying between theirs (31.778° and 32.944°). Likewise, Eq. (13.14)'s relative spread across eight bearings (`part4_near_center_universality`) falls from $\sim 11\%$ at $\delta = 5^\circ$ to $\sim 0.2\%$ at $\delta = 0.1^\circ$ at all three new centres (Gerizim: $11.08\% \rightarrow 0.22\%$), reproducing Section 13.8.5's Mecca figures ($14\% \rightarrow 0.14\%$ over a comparable range) to the same order of magnitude. **This settles, for three further concrete centres and not merely by the symbolic argument alone, that the pole singularity and the linear vanishing at $C/\bar{C}$ are properties of the oblique conformal construction itself, at any centre, not artefacts tied to Mecca's particular latitude.**

To Complete / Verify

Concretely open after this chapter: (i) a systematic atlas over a larger set of documented single-point orientation conventions (this chapter deliberately restricted itself to two, both textually and astronomically well-attested; other candidates considered and set aside for now include historical, rather than currently practised, qibla-like conventions, and would require separate historical verification of the kind given to the Islamic case in Chapter 4); (ii) the residual discrepancy between the 87.037° computed here and the $86^\circ 40'$ reported in the Lisbon source (Section 14.2) is not resolved, only bounded, for want of the source's own exact reference coordinates; (iii) whether the Jewish great-circle/rhumb-line dispute has, like the qibla's, an early astronomical-mathematical literature of its own predating the modern (post-Mercator) framing found in the consulted sources, not investigated here.

Chapter 15

Four-Axis Synthesis and Open Questions

Following the scheme already adopted elsewhere in the series to summarise concisely what is established, with what degree of certainty, and what remains open, this concluding chapter recaps the volume along four axes: mathematical, historical, computational, and epistemic.

15.1 Mathematical axis

- **Established with proof:** the generic non-coincidence between azimuthality and retroazimuthality with respect to the same centre ([Proposition 3.3](#)), an elementary but non-trivial consequence of the non-symmetry of spherical bearing; the exact quantification of that non-symmetry itself, via the signed spherical excess of the polar triangle P – Q –North Pole ([Proposition 2.4](#)), derived from Girard’s theorem and checked, independently of the derivation, against L’Huillier’s formula on 28 point pairs with zero residual to machine precision ([Section 2.2](#)); the analytic derivation of Craig’s formula ([Eq. \(9.1\)](#)) directly from the retroazimuthal condition plus the single design choice of straight equally-spaced meridians ([Proposition 9.3](#)), rather than its mere transcription and numerical check.
- **Established with machine-precision numerical verification:** the exact retroazimuthal property of Craig over the whole sphere ([Chapter 9](#)); that of Hammer on the front hemisphere, and, once the bearing read on the back hemisphere is corrected by the exact reflection formula $\theta_{\text{true}} = 180^\circ - \theta_{\text{read}}$ derived and verified in [Chapter 10](#) (initially conjectured, wrongly, as a plain $+180^\circ$ additive shift), over the whole sphere as well; that of Littrow at the single native equatorial point ([Chapter 8](#)); Hammer’s radial equidistance.
- **Established with proof, an item closed on both directions:** the retroazimuthal condition itself, recast as a vector-alignment criterion $(-x, -y) = k(\varphi, \lambda) (V_x, V_y)$ ([Eq. \(12.2\)](#)), is shown to unify Craig, Hammer, and Littrow as three different choices of the scalar field k acting on one common vector field V ([Chapter 12](#)), with the oblique conformal construction of [Chapter 13](#) added as a fourth cut, for which $k = \sec \varphi$ is proved necessary for every centre φ_C ([Proposition 12.4](#)). Whether $k = \sec \varphi$ is also *sufficient*, not merely necessary, within this family — left open at that chapter’s own close — is answered in the affirmative in [Proposition 12.5](#): substituting $k = \sec \varphi$ into [Eq. \(12.2\)](#) reproduces [Eq. \(13.5\)](#) term by term, symbolically, for every φ_C , closing the loop between the vector-field formalism of [Chapter 12](#) and the complex-analytic construction of [Chapter 13](#) — within this two-parameter family specifically, not as a claim about retroazimuthal constructions in general ([Proposition 12.5](#)’s own disclaimer).
- **Revisited by an original computation, with a result against expectations:** the uniqueness of Littrow as *the* conformal retroazimuthal projection ([Chapter 13](#)), repeated by Snyder (1993) but traced to Maurer (1919) as its actual origin ([Appendix B.3](#)) and not

independently re-verified, was taken up as a concrete task — render the heuristic rigidity argument rigorous — rather than left as a citation. The resulting analysis (Section 13.3) shows, symbolically, that the necessary local condition (harmonicity of the bearing function in isothermal coordinates) holds for *every* centre, not only the equatorial one, and finds, numerically, no monodromy obstruction either, for nine tested centres including Mecca's. An explicit numerical construction of a conformal retroazimuthal projection centred on Mecca is exhibited and checked independently on three fronts (retroazimuthal residual, Cauchy–Riemann residual, path-independence), all below 4×10^{-6} . This is reported as exactly what it is — a single verified numerical construction, not a closed-form disproof of the literature's uniqueness claim, which may rest on a stronger requirement this volume's formal definition does not capture (Section 13.3.5) — but it is enough to say that the heuristic argument, made rigorous, does *not* reproduce the expected conclusion.

- **Revisited a second time, with a further unexpected finding:** global injectivity of the Mecca construction (Section 13.4) was tested by a 358-point sampled search for image self-intersections, finding none, and by the already-known fact that the construction's Jacobian never vanishes away from $C, \bar{C}$ — together, evidence in favour of injectivity on the tested region, not a proof. The behaviour at the poles and at $\bar{C}$ asked for in the same item, however, produced a genuinely new result: both poles, ordinary points for the bearing function itself, turn out to be a second kind of singularity of the construction, with the reconstructed radius diverging directionally (numerically consistent with $r \sim e^\psi$) so that the map does not extend continuously to either pole at all, independently of the already-expected discontinuity at $\bar{C}$. The behaviour at $\bar{C}$ itself was checked along eight directions and initially showed a puzzling asymmetry: seven directions shrank cleanly towards $\bar{C}$ while an eighth diverged. Reconstructing $s = \ln r$ with the integration path reversed traced this to a crossing artefact — for that one direction, and only for it, the original path's fixed-longitude leg happens to pass directly through $\bar{C}$ — and, once diagnosed, the eighth direction falls cleanly into line with the other seven under the alternative path. The same two-path comparison, applied to the poles, found the two orders agreeing there to four or five significant figures, ruling out this particular artefact as an explanation and leaving the pole singularity as a genuine (though not yet proved necessary) feature of the construction.
- **Made exact by an independent, path-free method:** the numerical construction of the previous item is superseded, not merely supplemented, by a genuine closed form (Section 13.5): an auxiliary holomorphic function $g(\zeta)$ on the isothermal coordinate $\zeta = \lambda + i\psi$ yields x, y (Eq. (13.5)) with no numerical integration, no path convention, and no reference point, for every φ_C at once. This closed form settles, exactly, two items the numerical stage above could only leave as sampled evidence: (i) global injectivity fails everywhere, via an explicit fold — the map is literally a scaled sine of ζ (Section 13.6), so the classical non-injectivity of $\sin$ transplants directly to the sphere, proved analytically rather than searched for numerically; (ii) the pole singularity is intrinsic to the construction itself, not an artefact of the earlier path-integral method (settling the specific concern raised above), with an exact asymptotic rate $r \sim \frac{1}{2} \cos \varphi_C e^\psi$ (Proposition 13.14), an exact $O(\delta^2)$ anisotropic correction near C and $\bar{C}$ derived from the same g (Proposition 13.17), and, closing the item left open at that point, a direct geometric reading of the leading pole rate as a rescaling of the classical polar-stereographic radius (Proposition 13.18) — an elementary identity, not independently attributed to a specific source, confirmed here symbolically and numerically rather than merely observed to share the same order of growth.

- **Extended and independently re-checked:** the pole and $\bar{C}$ asymptotics of [Section 13.8.5](#), derived symbolically for arbitrary φ_C but numerically exercised there only at Mecca's own latitude, are checked again in [Section 14.4](#) at three further, independently motivated centres (Jerusalem, Bahjī, and Mount Gerizim, [Chapter 14](#)), reproducing the same residual pattern at all three to two or three significant figures — direct evidence, not only the symbolic argument, for the claim that neither singularity is specific to Mecca.
- **Revisited a third time, extending beyond Littrow:** whether the global-injectivity failure found for Littrow's family ([Section 13.6](#)) is peculiar to it, or shared by Craig and Hammer, was taken up in [Section 13.8](#). The literature search undertaken there found that Tobler (2002) already documents, in general terms for the whole retroazimuthal class, that such maps cannot cover the world without overlapping — resolving, at the qualitative level, whether the phenomenon is named in the practical literature. At the quantitative level, Craig was shown to admit its own explicit, symbolically verified closed-form fold ([Proposition 13.9](#)), structurally different from Littrow's ι (a latitude reflection at fixed longitude, rather than a longitude reflection at negated latitude) but equally exact; for Hammer, whose folding factor does not separate the way Craig's does, the natural candidate — the same fixed-longitude latitude reflection that produces Craig's fold — is now proved, not merely observed by failed search, to give something different: an exact mirror symmetry across the map's x -axis ([Proposition 13.11](#)), not a fold, because Hammer's unsigned y -factor is the sine/cosine companion of the very function z whose reflection symmetry the ansatz exploits. Whether some other curve folds Hammer was, at that point, still open; treating $z(\varphi, \lambda)$ directly, as anticipated, resolved it in [Section 13.8.5](#): reparametrising by distance z and bearing α from the centre, the plotted angle's rate of change with α is shown, in closed form, to have exactly the sign of $\cos \lambda$ ([Proposition 13.13](#)) — so it can only change sign at the front/back hemisphere boundary itself, never inside a hemisphere. Each hemisphere of Hammer's construction is therefore exactly, globally injective, upgrading the earlier sampled evidence into a proof, and explaining Hammer's own asymmetric sign convention as the exact correction for a genuine sign reversal in that rate of change, rather than an arbitrary choice.
- **Open:** direct access to Maurer (1919) — not Snyder (1993), which only reports the claim at second hand — to settle what the original uniqueness claim precisely asserts: a search located a plausible digitisation venue (the BSH digital library of *Annalen der Hydrographie*) but could not retrieve the specific volume, so the paper itself remains unobtained; and, more narrowly, whether Tobler's one-sentence remark ([Section 13.8.1](#)) is itself traceable to a more detailed primary source that would settle the exact multiplicity of the Craig and Hammer overlaps, not investigated here. The eighth-direction anomaly at $\bar{C}$ itself is no longer open, having been diagnosed and resolved as described above; nor, as of the closed-form construction above, are the former existence question for a non-equatorial closed form, or the path-artefact question for the pole singularity.

15.2 Historical axis

- **Established, on the basis of reliable secondary literature (King):** the essential chronology of the Islamic science of the qibla (mathematical geography from the eighth century, widely diffused approximate methods from the ninth, up to tabulated exact calculations); the dichotomy between sacred and mathematical geography; the dating and origin of the Craig (1909, Egypt) and Hammer (1910) projections.
- **Established independently, with original quantitative sources:** Ptolemy's systematic

error in the length of the Mediterranean (63° against $\sim 50^\circ$) and its correction by al-Khwarizmi, with a quantification of the effect of such an error on the qibla bearing computed with the exact formula (Chapter 6).

- **Revisited with an original derivation and quantitative check:** al-Battani’s approximate method (Chapter 7), reported by King (2019) only as a bare ratio with an unresolved subtraction-order ambiguity, is related here to a first-order tangent-plane expansion of the exact bearing formula, derived by hand and checked independently by finite differences; the expansion is then used to resolve the ambiguity by internal consistency rather than leave it open, and the resulting formula is scored against the exact bearing on all forty-four cities of Appendix A: mean absolute error 1.28° within 2000 km of Mecca, rising to 13.6° overall and 16.8° beyond 2000 km, with the omitted meridian-convergence factor ($\cos \varphi_C = 0.931$) accounting for only a modest share (a factor of 1.15) of that error relative to an idealised linear approximation. The qualitative literature description («accurate near Mecca, degrading with distance») is thus confirmed quantitatively, though al-Battani’s original text itself remains unconsulted (Chapter 7).
- **Traced to its primary sources, as far as available, and found more hedged than its secondary reports:** Tobler’s (2002) report of a possible Islamic precursor of the retroazimuthal projection, centuries earlier than Craig and Hammer, was first traced (Chapter 4) to the specific dispute behind it, King (1997) versus Dekker (2000), and then, in Chapter 5, consulted directly rather than through Tobler’s paraphrase or this volume’s own earlier secondary account. King’s own abstract turns out to hedge the antiquity claim with «likely», not to assert it flatly; Dekker’s own text narrows her disagreement with King to a specific rejection of a *documented, continuous* line of descent from ninth-century Baghdad, combined with an explicit agnosticism about the grids’ deeper origin, rather than a blanket denial of Islamic inspiration. Consulting Dekker directly also surfaced one previously unrecorded detail: a third, Schoy variant of the construction, with parallels bending away from the equator rather than toward it. Full access to either paper remains unobtained, so the item is narrowed and sharpened, not closed.
- **A second real single-point orientation dispute, identified and independently checked:** the Jewish great-circle/rhumb-line dispute over the direction of prayer toward Jerusalem (Section 14.2), textually distinct from but mathematically identical to the qibla’s own direction-facing problem, with one of its standard illustrative claims (the Lisbon «N86°40’E» paradox) reproduced independently to within 0.4° .

15.3 Computational axis

Every projection and every distortion analysis in this volume is accompanied by an executable, and executed, Python script (Appendix A lists the practical reference table drawn from them; the scripts themselves live in `code/`), whose numerical results (JSON files) are reported in the text without further rounding beyond what the script itself produces. No numerical value presented as «computed» in this volume was entered by hand or estimated: every table traces back explicitly to the script that generated it.

15.4 Epistemic axis: what sets this volume apart from the others in the series

Methodological Note

Unlike volumes 1–5 of the series, largely grounded in mathematical results proved in full within the volume itself (or traceable to standard proofs reproduced there), this volume confronts a different epistemic constraint more directly: much of the historical cartographic reference literature — Maurer (1919) in particular, the original source of the Littrow-uniqueness claim, of which Snyder (1993) is itself only a secondary report ([Appendix B.3](#)) — was not, at the time of writing, consultable in a full edition, but only through secondary sources (Wikipedia, reference implementations such as d3-geo-projection, academic abstracts). The volume has chosen, consistently with the discipline already adopted elsewhere in the author’s output, to flag explicitly every point at which this constraint leaves a claim in the state of «attributed but not independently re-verified» rather than presenting it as proved. This volume is thus, relative to the others in the series, closer to a carefully verified, annotated atlas of other people’s results accompanied by original numerical checks, than to a treatise of new theorems: a difference of genre stated explicitly from the preface, not a flaw to be silently corrected.

15.5 In one sentence

The retroazimuthal problem — born of a precise and ancient ritual need, the determination of the qibla — has no known *historical* construction achieving exactness, conformality, and an arbitrary centre together: Craig and Hammer choose global exactness at the cost of conformality; Littrow chooses conformality at the cost of freedom in the choice of centre. Whether this is an intrinsic limitation of the problem itself, however, no longer looks like the safe assumption it seemed at the outset of this volume: the attempt, in [Section 13.3](#), to make rigorous the standard heuristic for Littrow’s uniqueness instead produced a verified numerical construction of a conformal retroazimuthal projection centred on Mecca. Whether this reflects a genuine mathematical fact about the problem (that arbitrary centres are, in fact, achievable conformally, just not by any nineteenth- or twentieth-century closed-form construction) or a gap between this volume’s formal definition and whatever stronger property the historical uniqueness claim actually refers to, remains, honestly, the open question this volume closes on — a different, and more interesting, one than the question it opened with.

Appendix A

Practical Reference: Exact Qibla Bearings for World Cities

Methodological Note

This appendix replaces what was originally planned as a purely bibliographic code inventory with something of direct practical value: a ready-to-use reference table of exact qibla bearings, computed with the same verified formula used throughout this volume (Eq. (2.2)), for a broad, geographically diverse set of world cities. It also works through the calculation by hand for one city, step by step, so that the formula in Chapter 2 can be applied by a reader without a computer. Reproducibility details (which script generated which number) remain noted, briefly, in the caption of each table and in a short closing note; they are no longer the organising principle of the appendix.

A.1 Worked example: computing the qibla bearing by hand

Take Rome ($\varphi = 41.9028^\circ\text{N}$, $\lambda = 12.4964^\circ\text{E}$) and Mecca ($\varphi_M = 21.3891^\circ\text{N}$, $\lambda_M = 39.8579^\circ\text{E}$). Applying Eq. (2.2):

1. Longitude difference: $\Delta\lambda = \lambda_M - \lambda = 39.8579^\circ - 12.4964^\circ = 27.3615^\circ$.
2. Numerator: $\sin \Delta\lambda \cos \varphi_M = \sin(27.3615^\circ) \cos(21.3891^\circ) = 0.45964 \times 0.93094 = 0.42791$.
3. Denominator: $\cos \varphi \sin \varphi_M - \sin \varphi \cos \varphi_M \cos \Delta\lambda = \cos(41.9028^\circ) \sin(21.3891^\circ) - \sin(41.9028^\circ) \cos(21.3891^\circ) \cos(27.3615^\circ) = (0.74392)(0.36466) - (0.66822)(0.93094)(0.88827) = 0.27134 - 0.55251 = -0.28117$.
4. Bearing: $\theta = \text{atan2}(0.42791, -0.28117) = 123.28^\circ$ (measured clockwise from north; a positive numerator and negative denominator place the angle in the second quadrant, i.e. between east and south, consistent with a bearing a little south of east-southeast).

This matches, to the reported precision, the value for Rome in Table 6.1 (123.277°) and in Table A.1 below, confirming the internal consistency of every numerical table in this volume.

Key Result

A bearing of 123° is read on a compass as roughly «east-southeast», about 33° south of due east. A common intuitive mistake — drawing a straight line on a flat, equirectangular world map from Rome to Mecca and reading its compass direction off a protractor — gives a visibly different, and wrong, answer, because a straight line on such a map is a rhumb line (constant compass bearing), not a great circle, and the two coincide only along meridians and the equator. The whole point of a retroazimuthal projection (Chapters 8 to 10) is to make the *correct*, great-circle bearing directly readable with a ruler, without requiring the

reader to perform the trigonometric calculation above.

A.2 Reference table

Table A.1 lists, for forty-four cities spanning every inhabited continent, the exact qibla bearing (initial great-circle azimuth toward Mecca), the corresponding sixteen-point compass direction, and the great-circle distance to Mecca. Coordinates are nominal current WGS84 values for each city's commonly cited reference point (city centre), not independently surveyed for this volume; they are adequate for a reference table of this kind but are not a substitute for a locally surveyed qibla where sub-degree accuracy matters (e.g. for the orientation of a new building).

Methodological Note

Values computed by `code/qibla_world_cities_table.py` (Mecca reference point: 21.3891°N, 39.8579°E, the same value used throughout the volume), with great-circle distance computed using the mean Earth radius $R = 6371.0088$ km.

Table A.1: Exact qibla bearing and great-circle distance to Mecca for forty-four world cities.

City	Bearing (°)	Compass point	Distance (km)
Baghdad, Iraq	199.64	SSW	1398
Beijing, China	278.84	W	7385
Belluno, Italy	128.37	SE	3718
Berlin, Germany	136.67	SE	4135
Buenos Aires, Argentina	76.31	ENE	12062
Cairo, Egypt	136.13	SE	1292
Cape Town, South Africa	23.40	NNE	6556
Chicago, USA	48.67	NE	11151
Damascus, Syria	164.45	SSE	1393
Delhi, India	266.52	W	3833
Dhaka, Bangladesh	277.51	W	5170
Fez, Morocco	95.84	E	4594
Isfahan, Iran	225.89	SW	1711
Istanbul, Turkey	151.58	SSE	2410
Jakarta, Indonesia	295.13	WNW	7916
Jerusalem	157.11	SSE	1243
Kuala Lumpur, Malaysia	292.50	WNW	6970
Lahore, Pakistan	260.23	W	3596
London, United Kingdom	118.98	ESE	4799
Los Angeles, USA	23.84	NNE	13429
Madrid, Spain	103.98	ESE	4603
Mexico City, Mexico	46.60	NE	13651
Milan, Italy	123.56	ESE	3864
Mogadishu, Somalia	345.03	NNW	2231
Moscow, Russia	176.31	S	3826
Muscat, Oman	266.33	W	1920
New York, USA	58.48	ENE	10311
Palermo, Italy	118.70	ESE	3143

City	Bearing (°)	Compass point	Distance (km)
Paris, France	119.16	ESE	4501
Riyadh, Saudi Arabia	243.45	WSW	789
Rome, Italy	123.28	ESE	3424
Samarkand, Uzbekistan	239.73	WSW	3273
Sana'a, Yemen	326.32	NNW	810
Sao Paulo, Brazil	68.98	ENE	10604
Sarajevo, Bosnia and Herzegovina	134.75	SE	3185
Sydney, Australia	277.48	W	13232
Tehran, Iran	218.26	SW	1945
Timbuktu, Mali	76.48	ENE	4520
Tokyo, Japan	292.95	WNW	9471
Toronto, Canada	54.58	NE	10501
Tunis, Tunisia	112.68	ESE	3333
Venice, Italy	127.67	SE	3664
Xi'an, China	277.44	W	6816
Zanzibar City, Tanzania	1.32	N	3065

Key Result

The table makes concrete, with real numbers rather than a general statement, a fact that surprises most people unfamiliar with great-circle navigation: from New York the qibla bearing is 58.48° — east-northeast, not east or southeast as a flat-map intuition would suggest — because the shortest path from New York to Mecca on the sphere arcs up toward higher latitudes before curving back down. The same effect is even more pronounced from Los Angeles (23.84° , north-northeast) and from Chicago (48.67° , northeast). This is exactly the phenomenon that motivated Craig's and Hammer's retroazimuthal constructions: on an ordinary flat map, reading the qibla bearing with a protractor from a straight line to Mecca gives the rhumb-line bearing, not the great-circle one, and for cities at these longitudes and latitudes the two differ substantially.

To Complete / Verify

The table above uses city-centre coordinates as a practical approximation adequate for a reference table spanning distances of thousands of kilometres; it should not be used, without a proper local geodetic survey, to physically orient a specific building's qibla niche (*mihrab*), for which the coordinate precision required is well below the city-centre approximation used here, and for which the ellipsoidal (not purely spherical) geodetic problem would ordinarily be used for a rigorous result.

A.3 Full al-Battani comparison table

Chapter 7 evaluates al-Battani's approximate method (Eq. (7.3)) against the exact bearing for all forty-four cities of Table A.1, but reports there only a sixteen-city excerpt spanning the full distance range. The complete table, sorted by distance from Mecca as in Chapter 7's own discussion, is given here for reference; values are taken verbatim from `data/al_battani_approximation_analysis_results.json`, generated by `code/al_battani_approximation_analysis.py`.

Table A.2: Al-Battani's approximate bearing against the exact bearing, all forty-four cities, sorted by distance from Mecca.

City	Dist. (km)	Exact (°)	Al-Battani (°)	Error (°)
Riyadh, Saudi Arabia	789	243.45	243.96	+0.52
Sana'a, Yemen	810	326.32	324.23	−2.09
Jerusalem	1243	157.11	155.80	−1.31
Cairo, Egypt	1292	136.13	135.11	−1.02
Damascus, Syria	1393	164.45	163.44	−1.01
Baghdad, Iraq	1398	199.64	200.82	+1.18
Isfahan, Iran	1711	225.89	226.33	+0.44
Muscat, Oman	1920	266.33	263.13	−3.20
Tehran, Iran	1945	218.26	218.98	+0.73
Mogadishu, Somalia	2231	345.03	343.97	−1.06
Istanbul, Turkey	2410	151.58	150.66	−0.92
Zanzibar City, Tanzania	3065	1.32	1.42	+0.10
Palermo, Italy	3143	118.70	122.83	+4.13
Sarajevo, Bosnia and Herzegovina	3185	134.75	136.27	+1.52
Samarkand, Uzbekistan	3273	239.73	235.53	−4.21
Tunis, Tunisia	3333	112.68	118.23	+5.55
Rome, Italy	3424	123.28	127.32	+4.05
Lahore, Pakistan	3596	260.23	252.69	−7.53
Venice, Italy	3664	127.67	131.39	+3.73
Belluno, Italy	3718	128.37	132.06	+3.69
Moscow, Russia	3826	176.31	176.04	−0.27
Delhi, India	3833	266.52	258.29	−8.23
Milan, Italy	3864	123.56	128.65	+5.09
Berlin, Germany	4135	136.67	139.25	+2.58
Paris, France	4501	119.16	127.15	+7.99
Timbuktu, Mali	4520	76.48	83.24	+6.76
Fez, Morocco	4594	95.84	107.24	+11.40
Madrid, Spain	4603	103.98	115.32	+11.34
London, United Kingdom	4799	118.98	127.99	+9.00
Dhaka, Bangladesh	5170	277.51	266.87	−10.64
Cape Town, South Africa	6556	23.40	23.96	+0.56
Xi'an, China	6816	277.44	256.51	−20.93
Kuala Lumpur, Malaysia	6970	292.50	289.56	−2.95
Beijing, China	7385	278.84	251.92	−26.92
Jakarta, Indonesia	7916	295.13	296.72	+1.59
Tokyo, Japan	9471	292.95	255.94	−37.01
New York, USA	10311	58.48	109.89	+51.41
Toronto, Canada	10501	54.58	113.47	+58.89
Sao Paulo, Brazil	10604	68.98	54.71	−14.27
Chicago, USA	11151	48.67	113.80	+65.14
Buenos Aires, Argentina	12062	76.31	50.05	−26.26
Sydney, Australia	13232	277.48	311.42	+33.94
Los Angeles, USA	13429	23.84	120.45	+96.61
Mexico City, Mexico	13651	46.60	87.02	+40.42

Key Result

Reading down the error column confirms visually what [Chapter 7](#) reports as a summary statistic: the sign of the error is not random noise but tracks the geometry. For the nine nearest cities (through Istanbul, all within 2500 km) the error never exceeds 3.2° in magnitude and changes sign repeatedly; from Palermo onward (over 3000 km) it is consistently positive for every remaining Mediterranean and European city (Palermo through London), reflecting the systematic direction in which the meridian-convergence omission and the breakdown of linearity both push the bearing at these longitudes, before the pattern changes again for the most distant, non-Old-World cities.

Appendix B

Practical Reference: World-City Bearings to the Three Further Centres

Methodological Note

Appendix A gave a practical, ready-to-use table of exact qibla bearings for forty-four world cities. This appendix extends the same table, the same forty-four cities, and the same great-circle formula (Eq. (2.2)) to the three further centres of Chapter 14 — Jerusalem, Bahjí, and Mount Gerizim — and adds, for each, the rhumb-line bearing computed by the same method as Tables 14.1 and 14.2, so that the great-circle/rhumb-line divergence documented there for seven test localities can be inspected at the full, geographically diverse scale already established in Appendix A. Values computed by `code/world_cities_three_centres.py`, which reuses the city list of `code/qibla_world_cities_table.py` unchanged and the centre coordinates and formulas of `code/multi_center_atlas.py` unchanged; no new coordinates are introduced in this appendix.

Provenance note. The script that originally produced this table was lost before it could be archived in this project. It has since been rewritten from the unchanged city list and centre data above and re-run; all $44 \times 3 = 132$ rows were independently recomputed and checked character-for-character against the values already in this appendix, with an exact match throughout. The one place this re-verification did *not* confirm the existing text was the summary statistic below (smallest great-circle/rhumb-line divergence), which has been corrected.

B.1 Jerusalem

Reference point: the Temple Mount, 31.77806°N, 35.23583°E (Chapter 14).

Table B.1: Great-circle and rhumb-line bearing to Jerusalem, and great-circle distance, for forty-four world cities.

City	GC bearing (°)	Compass	Rhumb (°)	Distance (km)
Baghdad, Iraq	261.19	W	258.70	872
Beijing, China	290.87	WNW	262.95	7120
Belluno, Italy	120.94	ESE	128.91	2535
Berlin, Germany	134.05	SE	142.44	2901
Buenos Aires, Argentina	64.59	ENE	53.03	12240
Cairo, Egypt	62.18	ENE	63.20	427
Cape Town, South Africa	15.42	NNE	13.57	7515
Chicago, USA	45.57	NE	95.88	9956

City	GC bearing (°)	Compass	Rhumb (°)	Distance (km)
Damascus, Syria	207.07	SSW	206.78	216
Delhi, India	285.68	WNW	274.99	4025
Dhaka, Bangladesh	292.27	WNW	279.28	5443
Fez, Morocco	82.40	E	93.82	3741
Isfahan, Iran	270.82	W	266.39	1547
Istanbul, Turkey	149.46	SSE	151.43	1168
Jakarta, Indonesia	304.41	NW	299.04	8662
Jerusalem	62.58	ENE	62.58	2
Kuala Lumpur, Malaysia	303.06	WNW	294.59	7607
Lahore, Pakistan	280.94	W	270.39	3681
London, United Kingdom	113.59	ESE	127.10	3610
Los Angeles, USA	23.76	NNE	91.01	12186
Madrid, Spain	92.97	E	105.38	3597
Mexico City, Mexico	41.19	NE	84.17	12528
Milan, Italy	115.13	ESE	124.07	2707
Mogadishu, Somalia	343.31	NNW	342.25	3471
Moscow, Russia	184.98	S	184.02	2673
Muscat, Oman	296.93	WNW	291.79	2450
New York, USA	54.06	NE	95.80	9170
Palermo, Italy	102.91	ESE	109.48	2108
Paris, France	112.49	ESE	124.51	3333
Riyadh, Saudi Arabia	307.64	NW	305.06	1367
Rome, Italy	111.83	ESE	119.16	2305
Samarkand, Uzbekistan	262.93	W	253.04	2978
Sana'a, Yemen	335.11	NNW	333.53	2038
Sao Paulo, Brazil	57.76	ENE	54.78	10647
Sarajevo, Bosnia and Herzegovina	126.80	SE	132.38	1990
Sydney, Australia	286.73	WNW	300.96	14128
Tehran, Iran	258.38	WSW	253.76	1554
Timbuktu, Mali	59.36	ENE	66.62	4192
Tokyo, Japan	303.80	NW	267.43	9143
Toronto, Canada	51.01	NE	97.49	9323
Tunis, Tunisia	96.30	E	103.66	2361
Venice, Italy	119.64	ESE	127.49	2490
Xi'an, China	290.20	WNW	267.62	6715
Zanzibar City, Tanzania	354.53	N	354.30	4240

B.2 Bahjí

Reference point: the Shrine of Baha'u'llah, 32.94361°N, 35.09194°E ([Chapter 14](#)).

Table B.2: Great-circle and rhumb-line bearing to Bahjí, and great-circle distance, for forty-four world cities.

City	GC bearing (°)	Compass	Rhumb (°)	Distance (km)
Baghdad, Iraq	269.80	W	267.26	864

City	GC bearing (°)	Compass	Rhumb (°)	Distance (km)
Beijing, China	292.04	WNW	263.92	7062
Belluno, Italy	118.98	ESE	126.94	2435
Berlin, Germany	132.87	SE	141.24	2784
Buenos Aires, Argentina	63.44	ENE	52.45	12290
Cairo, Egypt	47.60	NE	48.59	487
Cape Town, South Africa	14.97	NNE	13.21	7637
Chicago, USA	44.93	NE	95.25	9847
Damascus, Syria	240.41	WSW	240.08	127
Delhi, India	287.67	WNW	276.83	4026
Dhaka, Bangladesh	293.83	WNW	280.65	5446
Fez, Morocco	80.41	E	91.87	3696
Isfahan, Iran	275.69	W	271.19	1548
Istanbul, Turkey	146.91	SSE	148.85	1048
Jakarta, Indonesia	305.53	NW	299.83	8709
Jerusalem	355.03	N	355.00	131
Kuala Lumpur, Malaysia	304.27	NW	295.51	7643
Lahore, Pakistan	283.08	WNW	272.40	3673
London, United Kingdom	112.26	ESE	125.76	3506
Los Angeles, USA	23.39	NNE	90.50	12062
Madrid, Spain	91.09	E	103.53	3528
Mexico City, Mexico	40.36	NE	83.58	12429
Milan, Italy	113.14	ESE	122.08	2613
Mogadishu, Somalia	343.83	NNW	342.72	3597
Moscow, Russia	185.46	S	184.44	2544
Muscat, Oman	299.77	WNW	294.54	2502
New York, USA	53.29	NE	95.09	9070
Palermo, Italy	99.74	E	106.31	2043
Paris, France	110.95	ESE	122.95	3233
Riyadh, Saudi Arabia	311.73	NW	309.08	1451
Rome, Italy	109.27	ESE	116.58	2221
Samarkand, Uzbekistan	265.40	W	255.41	2936
Sana'a, Yemen	336.47	NNW	334.84	2159
Sao Paulo, Brazil	56.64	ENE	54.10	10688
Sarajevo, Bosnia and Herzegovina	124.39	SE	129.95	1887
Sydney, Australia	288.06	WNW	301.43	14186
Tehran, Iran	263.16	W	258.47	1526
Timbuktu, Mali	57.47	ENE	64.82	4213
Tokyo, Japan	304.81	NW	268.19	9075
Toronto, Canada	50.33	NE	96.82	9217
Tunis, Tunisia	93.33	E	100.71	2306
Venice, Italy	117.59	ESE	125.42	2391
Xi'an, China	291.49	WNW	268.70	6675
Zanzibar City, Tanzania	354.55	N	354.29	4370

B.3 Mount Gerizim

Reference point: the Samaritan holy summit, 32.200861°N, 35.273258°E (Chapter 14).

Table B.3: Great-circle and rhumb-line bearing to Mount Gerizim, and great-circle distance, for forty-four world cities.

City	GC bearing (°)	Compass	Rhumb (°)	Distance (km)
Baghdad, Iraq	264.19	W	261.71	859
Beijing, China	291.25	WNW	263.29	7091
Belluno, Italy	120.10	ESE	128.11	2505
Berlin, Germany	133.48	SE	141.90	2863
Buenos Aires, Argentina	64.21	ENE	52.85	12265
Cairo, Egypt	57.01	ENE	58.04	453
Cape Town, South Africa	15.34	NNE	13.51	7561
Chicago, USA	45.28	NE	95.65	9922
Damascus, Syria	212.97	SSW	212.69	173
Delhi, India	286.39	WNW	275.67	4016
Dhaka, Bangladesh	292.83	WNW	279.79	5435
Fez, Morocco	81.65	E	93.11	3732
Isfahan, Iran	272.55	W	268.12	1538
Istanbul, Turkey	148.22	SSE	150.21	1128
Jakarta, Indonesia	304.84	NW	299.36	8671
Jerusalem	6.65	N	6.66	48
Kuala Lumpur, Malaysia	303.51	WNW	294.95	7612
Lahore, Pakistan	281.69	WNW	271.12	3669
London, United Kingdom	113.01	ESE	126.55	3578
Los Angeles, USA	23.55	NNE	90.83	12144
Madrid, Spain	92.24	E	104.68	3579
Mexico City, Mexico	40.83	NE	83.96	12498
Milan, Italy	114.30	ESE	123.27	2679
Mogadishu, Somalia	343.64	NNW	342.57	3514
Moscow, Russia	184.96	S	184.01	2626
Muscat, Oman	298.03	WNW	292.88	2460
New York, USA	53.73	NE	95.54	9140
Palermo, Italy	101.69	ESE	108.29	2092
Paris, France	111.84	ESE	123.88	3302
Riyadh, Saudi Arabia	309.38	NW	306.79	1390
Rome, Italy	110.80	ESE	118.15	2281
Samarkand, Uzbekistan	263.75	W	253.85	2955
Sana'a, Yemen	335.83	NNW	334.25	2078
Sao Paulo, Brazil	57.38	ENE	54.56	10669
Sarajevo, Bosnia and Herzegovina	125.77	SE	131.38	1958
Sydney, Australia	287.25	WNW	301.15	14141
Tehran, Iran	259.99	W	255.37	1534
Timbuktu, Mali	58.71	ENE	66.01	4207
Tokyo, Japan	304.12	NW	267.70	9112
Toronto, Canada	50.71	NE	97.24	9290
Tunis, Tunisia	95.17	E	102.56	2348
Venice, Italy	118.78	ESE	126.65	2460
Xi'an, China	290.63	WNW	268.01	6693
Zanzibar City, Tanzania	354.66	N	354.42	4286

Key Result

Across all forty-four cities and all three centres (excluding Jerusalem's own trivially near-zero distance to itself), the great-circle and rhumb-line bearings never coincide exactly: the smallest divergence found is Jerusalem's own bearing to Mount Gerizim, 0.02° at 48 km — smaller than Jerusalem's bearing to Bahjí, 0.03° at 131 km, the next-closest case — and the largest is Los Angeles' bearing to Mount Gerizim, 67.28° at 12144 km. This reproduces at full geographic scale the same qualitative pattern already established on the seven-city test set of [Chapter 14](#): the divergence grows, in a broad if not perfectly monotonic way, with distance from the centre and with the difference in latitude between city and centre, exactly as the spherical-triangle geometry of [Chapter 2](#) predicts.

References

Methodological Note

Not in BibTeX format for this edition (a short, self-contained volume): references are cited in the text by author and year, or by title, and pointed here to the source actually consulted. Where the consulted source is secondary relative to the work cited (e.g. an abstract rather than the full text), this is specified.

- Craig, J. (1910). *Map Projections*. Cairo: Ministry of Finance, Survey Department. [Primary work not consulted directly; biographical data and publication circumstances from cartographic encyclopaedic entries and from Wikipedia, entry *James Ireland Craig*.]
- Hammer, E. (1910). «Gegenazimutale Projektionen». *Petermanns Mitteilungen*, 56(3): 153–155. [Not consulted directly; cited via Tobler 2002 and cartographic encyclopaedic entries.]
- Kennedy, E. S., Kennedy, M. H. (1987). *Geographical Coordinates of Localities from Islamic Sources*. Frankfurt: Institut für Geschichte der Arabisch-Islamischen Wissenschaften. [Consulted indirectly via Mercier, *e-Perimetron* 15(2).]
- Dekker, E. (2000). «Cartographic Grids from Iran: An Early Version of the Retro-Azimuthal Orthographic Projection?». *The Cartographical Journal*, 37: 109–116. [Initially consulted only as an abstract and via a summary of its reception (King’s own annotated publication list); cf. [Chapter 4](#). Consulted again directly, via a scholarly aggregator’s cached preview including the abstract, the concluding paragraph, and two further technical passages (still short of the full text), in [Chapter 5](#), which narrows and corrects the first-pass secondary account of Dekker’s own position.]
- Jackson, J. E. (1968). Source of the retro-azimuthal orthographic construction discussed by Dekker (2000). [Not consulted directly; cited via Dekker’s abstract.]
- King, D. A. (1993). *Astronomy in the Service of Islam*. Aldershot: Variorum. [Consulted as an abstract/summary via Academia.edu.]
- King, D. A. (1997). «Two Iranian World Maps for Finding the Direction and Distance to Mecca». *Imago Mundi*, 49: 62–82. [Initially identified only via Dekker (2000) and via a summary of King’s own publication list, its role in the Tobler (2002) precursor claim clarified only at second hand; cf. [Chapter 4](#). Consulted again directly, via the journal’s own platform (Taylor & Francis Online), for its own published abstract in [Chapter 5](#); still short of the full article text.]
- Schoy, C. Author of a further retroazimuthal-type construction, distinct from Craig’s and Hammer’s, in which parallels bend away from rather than toward the equator. [Not consulted directly at any point; known only via a passing mention in Dekker (2000) above, consulted through the same cached excerpts; neither the specific publication nor its date has been identified or verified independently; cf. [Section 5.3](#).]

- King, D. A. (2019). «The enigmatic orientation of the Grand Mosque of Cordoba». [Consulted via search-engine-indexed excerpts (ResearchGate); source of the explicit al-Battani qibla formula and its geometric construction quoted in [Chapter 4](#).]
- King, D. A. (1996). «Astronomy in Islamic society: qibla, gnomonics and prayer-times». In *Encyclopedia of the History of Arabic Science*. [Consulted as an abstract/summary via Academia.edu.]
- King, D. A. (1999). *World-Maps for Finding the Direction and Distance to Mecca: Innovation and Tradition in Islamic Science*. Leiden: Brill & London: Al-Furqan Islamic Heritage Foundation. [Not consulted directly; cited via the author's own bibliography and via King 2020.]
- King, D. A. (2020). «Islamic sacred geography: finding the qibla by the sun and stars». [Consulted as an abstract/summary via Academia.edu.]
- Mercier, R. «Meridians of reference and mathematical geography in the medieval Muslim West». *e-Perimetron*, 15(2). [Consulted for the quantitative analysis of the Ptolemaic-Islamic longitude error.]
- Snyder, J. P. (1987). *Map Projections: A Working Manual*. USGS Professional Paper 1395. Washington, DC: US Government Printing Office. [Not consulted in full edition for this volume; cited via secondary sources that report it as the primary reference for the formulas.]
- Snyder, J. P. (1993). *Flattening the Earth: Two Thousand Years of Map Projections*. Chicago: University of Chicago Press. [Not consulted in full edition for this volume; pp. 227–229 (Craig, Hammer) and p. 135 (Littrow) cited via secondary sources. A renewed search for the full text (bookseller listings, Google Books preview, academic repositories) found only reviews, tables of contents and short excerpts, not the passages needed to settle the original uniqueness-claim question raised in [Chapter 13](#). That search did, however, establish that Snyder's own uniqueness statement is itself secondary: it traces to Maurer (1919) below, which is now the item actually needed and still unobtained.]
- Maurer, H. (1919). «Das winkeltreue gegenazimutale Kartennetz nach Littrow (Wiers Azimutdiagramm)». *Annalen der Hydrographie und maritimen Meteorologie*, 47(1–2): 14–22. [Identified, via a citation trail through Tobler (2002) below and standard cartographic bibliographies, as the likely original source of the Littrow-uniqueness claim discussed throughout [Chapter 13](#); the German-language paper itself has not been obtained or consulted directly. Two earlier papers by the same author, Maurer (1911) «Gegenazimutale Projektionen», *Kartographischer Monatsbericht IV in Petermanns Mitteilungen* 57(5): 255–256, and Maurer (1914) «Die Definitionen in der Kartenentwurfslehre...», *Petermanns Geographische Mitteilungen* 60(2): 61–67, 116–121, appear in the same bibliographic trail and may contain related material; neither has been consulted either.]
- Hinks, A. R. (1929). «A Retro-Azimuthal Equidistant Projection of the Whole Sphere». *Geographical Journal*. [Named, via Tobler (2002) below, as the author of a further retroazimuthal construction and of a related historical Rugby-centred chart; not consulted directly. Mentioned in the open *todobox* of [Chapter 12](#) as a variant not examined in this volume.]
- Snyder, J. P., Voxland, P. M. (1989). *An Album of Map Projections*. USGS Professional Paper 1453. Denver: USGS. [Not consulted in full edition; cited via *Directory of Map Projections*, mapmathematics.com.]

- Bugayevskiy, L. M., Snyder, J. P. (1995). *Map Projections: A Reference Manual*. Bristol: Taylor and Francis. [Not consulted in full edition; p. 133 cited via Wikipedia.]
- Tobler, W. (2002). «Qibla, and Related, Map Projections». *Cartography and Geographic Information Science*, 29(1): 17–23. doi:10.1559/152304002782064574. [Consulted as an abstract via ResearchGate and via secondary summaries (UC Santa Barbara Geography, Gale Academic OneFile); its «recent discovery from Iran» remark identified and traced to King (1997) and Dekker (2000), see Chapter 4; full article text still not consulted directly. The same abstract’s general remark that retroazimuthal parallels bend toward the equator, so that no member of the family can display the whole world without the map overlapping itself, is used in Sections 13.8.1 and 13.8.2 of Chapter 13 as literature confirmation of the fold/injectivity-failure phenomenon derived there in closed form.]
- d3-geo-projection, open-source repository. `src/craig.js`, `src/hammerRetroazimuthal.js`, `src/littrow.js`. <https://github.com/d3/d3-geo-projection>. [Reference implementation used to transcribe the projection formulas used in this volume; consulted directly, source code quoted in the text.]
- Wikipedia, entries: *Craig retroazimuthal projection*, *Hammer retroazimuthal projection*, *Littrow projection*, *James Ireland Craig*. [Consulted for biographical and historical context; the formulas were verified by cross-checking against d3-geo-projection, not taken from Wikipedia alone.]
- Levin, N. (2002). «The Direction-Facing Problem in Religion and Geography». [Consulted directly, PDF via Rutgers University; used in Chapter 14 for the general framing of the great-circle/rhumb-line direction-facing problem across religious traditions, and for the Daniel 6:11 / post-70 CE codification account of Jewish prayer orientation.]
- Torah Musings (blog), articles «Facing Jerusalem» (2011), «Orientation During Prayer» (2012), and «Orientation Revisited» (2012). [Consulted directly; secondary source, not a primary halakhic text, for the Levush/Emunas Chachamim/Netzach Yisrael/Mor Uketziah dispute over great-circle versus rhumb-line orientation toward Jerusalem (Chapter 14); primary sources (Levush, *Orach Chaim* 94:3; R. Yisrael of Zamosc, *Netzach Yisrael*; R. Yakov Emden, *Mor Uketziah* 150) not consulted directly.]
- KosherJava.com, pages «Rhumb Line» and «Qiblih.com – Find the direction to Bahji» (qiblih.com). [Consulted directly; corroborating secondary sources for the same Jewish-orientation dispute and for the Baha’i Qiblih’s explicit specification of the great-circle («shorter arc») convention, Chapter 14.]
- Wikipedia, entries: *Temple Mount*, *Qiblih*, *Shrine of Baha’u’llah*, *Mount Gerizim*. [Consulted directly for the coordinates of the additional centres used in Chapter 14 (31.77806°N, 35.23583°E for the Temple Mount; 32.94361°N, 35.09194°E for the Shrine of Baha’u’llah at Bahji; 32.200861°N, 35.273258°E for the summit of Mount Gerizim) and for the Baha’i Qiblih’s official specification of great-circle bearing.]
- the-samaritans.net, page «Mount Gerizim and the Samaritans». [Consulted directly; source for Mount Gerizim’s status as one of the Samaritan community’s five articles of faith and for the orientation of Samaritan prayer toward it, Chapter 14.]
- Schorch, S. (2018). Discussion of a Samaritan-Arabic prayer by the theologian Abu l-Hasan as-Suri, published via the Institute for Advanced Study. [Consulted directly; source for the textual parallel between this prayer and the Islamic qibla formula, and for the historical entanglement between the Samaritan Gerizim-orientation and the Jewish Jerusalem-orientation, Chapter 14.]